Perturbative calculation of quasinormal modes of d-dimensional black holes

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ABSTRACT: We study analytically quasinormal modes in a wide variety of black hole spacetimes, including d-dimensional asymptotically flat spacetimes and non-asymptotically flat spacetimes (particular attention has been paid to the four dimensional cases). We extend the analytical calculation to include first-order corrections to analytical expressions for quasinormal mode frequencies by making use of a monodromy technique. All possible type perturbations are included in this paper. The calculation performed in this paper shows that systematic expansions for uncharged black holes include different corrections with the ones for charged black holes. This difference makes them have a different n-dependence relation in the first-order correction formulae. The method applied above in calculating the first-order corrections of quasinormal mode frequencies seems to be unavailable for black holes with small charge. This result supports the Neitzke's prediction. On what concerns quantum gravity we confirm the view that the $\ln 3$ in $d = 4$ Schwarzschild seems to be nothing but some numerical coincidences.

KEYWORDS: Black Holes, Models of Quantum Gravity, Classical Theories of Gravity.

Contents

1. Introduction

The stability of black holes has been discussed since researchers found that black holes were shown to radiate and evaporate when we add in the ideas of quantum mechanics to them [1, 2] and hence people tried to make sure whether the black hole solution under consideration was really a stable one of the *classical* theory. The pioneering work on this problem was carried by Regge and Wheeler [3], who focused on analyzing the linear stability of four dimensional Schwarzschild black hole. They found that one can use a Schrödinger-like equation to describe the linear perturbations. This work was latter extended to many other black hole solutions and is now known as the quasinormal modes (QNMs) which can be described as a "characteristic sound" of black holes (A lot of investigation have been made on this subject [4 – 7]). QNMs are excited by the external perturbations(may be induced, for example, by the falling matter). They appear as damped oscillations described by the complex characteristic frequencies which are entirely fixed by the parameters of the given black hole spacetime, and independent of the initial perturbation [8]. These frequencies can be detected by observing the gravitational wave signal [9]: this makes QNMs be of particular relevance in gravitational wave astronomy.

Although the QNMs are important in the observational aspects of gravitational waves phenomena mentioned above, there is suggestion that the asymptotic QNMs may find a very important place in Loop Quantum Gravity (LQG). Recently Hod [10] made an interesting proposal to infer quantum properties of black holes from their classical oscillation spectrum . The idea was based on the Bekenstein's conjecture [11] that in a quantum theory

of gravity the surface area of a non-extremal black hole should have a discrete eigenvalue spectrum. The eigenvalues of this spectrum are likely to be uniformly spaced. According to the numerical values computed by Nollert [4] and later confirmed by Andersson [12], Hod observed that the real parts of the asymptotic form of high overtones of a Schwarzschild black hole can be written as:

$$\frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3,$$

where T_H is the Hawking temperature. Using this conjecture and Bekenstein's conjecture, he obtained the Bekenstein-Hawking entropy for the Schwarzschild black hole. He also showed that this approach is compatible with the statistical mechanical interpretation of black hole entropy. Later, Dreyer [13] noted that this is helpful for the calculation of the so called Barbero-Immirzi parameter, a free parameter introduced as the Barbero-Immirzi connection in the calculation of LQG. The only way so far that could be used to fix this parameter comes from black hole entropy. Dreyer use the conjecture made by Hod, he fixed the value for the Immirzi parameter by $\frac{\ln 3}{2\pi\sqrt{2}}$. Using this value, he suggested that "the appropriate gauge group of quantum gravity is SO (3) and not its covering group SU (2)". However, more recently Corichi [14] argued that the LQG allows us to keep SU (2) as the gauge group, and at the same time have a consistent description with the results of Dreyer. He reconsidered the physical process that would give rise to the quasinormal frequency (QN frequency) as mentioned in [13]: an appearance or disappearance of a puncture with spin $j_{\min}$. Taking into account of the *local* conservation of fermion number, Corichi obtained that "the minimum allowed value for the 'spin' of the resulting free edge is $j_{\min} = 1$": this agrees with the results of Dreyer. However, all this problems are far from being resolved.

As mentioned in the above paragraph, the Hod's conjecture was based on the numerical results evaluated by Nollert [4]. People may think this agreement is just a coincidence. Motl [15, 16], however, confirmed analytically Nollert's result by two different methods. In ref. [15], the author used Nollert's continued fraction expansion for the 4-dimensional Schwarzschild and showed that the asymptotic QN frequencies are in good agreement with Hod's result. The monodromy technique was first introduced in [16] to analytically compute asymptotic QN frequencies and later extended in [17], so that it can also be used in the computation of d-dimensional asymptotically dS, AdS spacetime. A question, however, produced as showed in [16, 17] is that the suggestion in [10, 13] was proved not to be universal and be only applicable to the Schwarzschild solution. It is necessary to stress that even if the ideas in [10, 13] turn out not to be universal, it is still the case that QN frequencies will play some role in the realm of quantum gravity, since the LQG is far from being successful, and recent studies show that they have interpretation in conformal field theory through AdS/CFT correspondence [18–21]. Moreover, the possible appearance of ln 2 in the asymptotic frequencies [16] could support the claims [14] that the gauge group of LQG should be SU(2) despite the ln 3 for Schwarzschild.

It is important to note that the behavior of QNMs for the lowest modes (frequencies with a smaller imaginary part) is totally different from the one for the high overtone. We

have shown that the real part of the QN frequencies approaches some certain value as the imaginary part approaches infinity. There has no such phenomena for the low-lying modes. A question of particular relevance that immediately follows is what will happen for the QNMs when the imaginary part has a middle value between these two extreme cases, and why these two cases behavior so differently (the resolution of this problem is helpful to deduce analytically the asymptotic value of the QN frequencies). Recently Musiri and Siopsis [22, 23] have studied in detail about this question for Schwarzschild in asymptotically flat and asymptotically AdS spacetime. Extending the technique introduced in [16] to obtain a systematic expansion including corrections in $1/\sqrt{\omega}$, they obtained the j and l dependence of the first correction for arbitrary j. Their results are in good agreement with the results obtained by numerical methods in the case of scalar and gravitational waves. However, they have discussed nothing about more general spacetime background, such as the case of Reissner-Nordström (RN) black holes and higher dimensions. The main purpose of this paper is to study analytically the first-order correction to the asymptotic form of QN frequencies for more general sapcetime background. In this work we shall make use of the remarkable results obtained by Ishibashi and Kodama (we refer the reader to [24 – 26] for detail on this subject). They studied in detail the perturbation theory of static, spherically symmetric black holes in any space-time dimension $d > 3$ and allowing for the possibilities of both electromagnetic charge and a background cosmological constant. According to them, the perturbations come in three types: tensor type perturbations, vector type perturbations and scalar type perturbations, and linear perturbations in d-dimensions can be described by a set of equations which may be denoted as the Ishibashi-Kodama (IK) master equations.

The organization of this paper is as follows. In next section we apply the monodromy method introduced in [16] to the IK master equations expanded near the several singularities in the complex plane to analytically compute both zeroth-order and first-order asymptotic quasinormal frequencies for static, spherically symmetric black hole spacetimes in dimension $d > 3$. Section 3 is the last section of the paper, where we have a discussion about our results, listing some problems encountered in this paper. Some future directions are also included in this section. The last section is our appendices. In appendix A we make use of expanding the tortoise coordinate to first order at the singularities in the spacetimes considered in this paper, providing a full analysis of the potentials at several singularities in the complex plane, and obtaining a list of first-order IK master equation potentials.

2. Perturbative calculation of quasinormal modes

In this section we first review the perturbation theory roughly for spherically symmetric, static d-dimensional black holes ($d > 3$), with mass M, charge Q and background cosmological constant Λ, and the computation of QNMs and QN frequencies. We refer the reader to [17] for more detail.

For a massless, uncharged, scalar field, Ψ, after a harmonic decomposition of the scalar field as $\Psi = \sum_{\ell,m} r^{\frac{2-d}{2}} \psi_\ell(r,t) Y_{\ell m}(\theta_i)$, where the θ_i are the $(d-2)$ angles and the $Y_{\ell m}(\theta_i)$

are the d-dimensional spherical harmonics, and a Fourier decomposition of the scalar field $\psi_\ell(r,t) = \Psi(r)e^{i\omega t}$, the wave equation can be decoupled as a Schrödinger-like equation

$$-\frac{d^2\Psi(r_*)}{dr_*^2} + V(r_*)\Psi(r_*) = \omega^2\Psi(r_*), \tag{2.1}$$

where r_* is tortoise coordinate defined as $dr_* = \frac{dr}{f(r)}$ and $V(r_*)$ is the potential, both determined from the function $f(r)$ in the background metric. The potential $V(r_*)$ depends on the background space-time metric and the perturbative type (appendix A). For QNMs we need some boundary conditions, so that

$$\Psi(r_*) \sim e^{i\omega r_*} \text{ as } r_* \to -\infty,$$
$$\Psi(r_*) \sim e^{-i\omega r_*} \text{ as } r_* \to +\infty.$$

Using these boundary conditions and the monodromy technique, we shall show how to calculate the asymptotic QN frequencies and their first-order correction in all static, spherically symmetric black hole spacetimes (including asymptotically flat spacetimes and non-asymptotically flat spacetimes). As an example, we may pay more attention to the case of $d = 4$. For some cases, we shall list some corrected QN frequencies, so that we can show it is a reasonable correction by comparing with the numerical results.

2.1 The Schwarzschild case

Although the perturbative calculation for the Schwarzschild solution in four-dimension and higher dimension have been discussed in [22] and [27], respectively, we first review it roughly for completeness.

For Schwarzschild black hole, we have

$$f(r) = 1 - \frac{2m}{r^{d-3}},$$

with the roots

$$r_n = \left|(2m)^{\frac{1}{d-3}}\right| \exp\left(\frac{2\pi i}{d-3}n\right), \quad n = 0, 1, \ldots, d-4.$$

The radial wave equation for gravitational perturbations in the black-hole background can be written as

$$-\frac{d^2\Psi(r_*)}{dr_*^2} + V[r(r_*)]\Psi(r_*) = \omega^2\Psi(r_*)$$

in the complex r-plane.

As mentioned above, the boundary conditions are

$$\Psi(r_*) \sim e^{\mp i\omega r_*} \text{ as } r_* \to \pm\infty,$$

assuming Re $\omega > 0$. Then we obtain

$$F(r_*) \sim 1 \text{ as } r_* \to +\infty,$$
$$F(r_*) \sim e^{2i\omega r_*} \text{ as } r_* \to -\infty, \tag{2.2}$$

if we rewrite the QNMs as $F(r_*) = e^{i\omega r_*}\Psi(r_*)$. The clockwise monodromy of $F(r_*)$ around the $r = r_H$ can be easily obtained by continuing the coordinate r analytically into the complex plane, *i.e.*,

$$\mathcal{M}(r_H) = e^{\frac{2\pi\omega}{k_H}},$$

where $k_H = \frac{1}{2}f'(r_H)$ is the surface gravity at the horizon.

Near the black hole singularity ($r \sim 0$), the tortoise coordinate may be expanded as

$$r_* = \int \frac{dr}{f(r)} = -\frac{1}{d-2}\frac{r^{d-2}}{2m} - \frac{1}{2d-5}\frac{r^{2d-5}}{(2m)^2} + \cdots,$$

where $f(r) = 1 - \frac{2m}{r^{d-3}}$ and m is the mass of the black hole[1]. When we define $z = \omega r_*$, the potential near the black hole singularity for the three different type perturbations can be expanded, respectively, as (appendix A)

$$V[z] \sim -\frac{\omega^2}{4z^2}\left\{1 - j^2 - W(j)\left(\frac{z}{\omega}\right)^{(d-3)/(d-2)} + \cdots\right\}, \tag{2.3}$$

where

$$W(j) = \begin{cases} W_{ST} & j = 0, \\ W_{SV} & j = 2, \\ W_{SS} & j = 0, \end{cases}$$

and the explicit expressions of W_{ST}, W_{SV} and W_{SS} can be found in appendix A. Then the Schrödinger-like wave equation (2.1) with the potential (2.3) wave equation can be depicted as

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}}\mathcal{H}_1\right)\Psi = 0, \tag{2.4}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined as

$$\mathcal{H}_0 = \frac{d^2}{dz^2} + \left[\frac{1-j^2}{4z^2} + 1\right], \quad \mathcal{H}_1 = -\frac{W(j)}{4}z^{-\frac{d-1}{d-2}}.$$

Taking into account of $\omega \to \infty$, the zeroth-order wave equation becomes

$$\mathcal{H}_0\Psi^{(0)} = 0,$$

with general solutions in the form of

$$\Psi^{(0)} = A_+ J_{j/2}(z) + A_- J_{-j/2}(z), \tag{2.5}$$

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[1]In fact, one can relate it with the ADM mass M by

$$M = \frac{(d-2)\mathcal{A}_{d-2}}{8\pi G_d}m,$$

where $\mathcal{A}_n$ is the area of an unit n-sphere, $\mathcal{A}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$.

where and below $J_\nu(x)$ represents a Bessel function of the first kind. According to the boundary conditions (2.2), one can define

$$F^{(0)}(z) = F_+^{(0)}(z) - e^{-\pi j i/2} F_-^{(0)}(z),$$

which approaches $-e^{-i\alpha_+} \sin \frac{j\pi}{2}$ as $z \to +\infty$, where $\alpha_\pm = \frac{(1\pm j)\pi}{4}$. Considering the behavior of the wave function as $z \to -\infty$, we may deduce that

$$\mathcal{M}(r_H) \sim -\frac{\sin 3j\pi/2}{\sin j\pi/2},$$

leading to the generic expression of d-dimensional QN frequencies as showed in [17]

$$\frac{\omega}{T_H} = (2n+1)\pi i + \ln(1 + 2\cos j\pi),$$

where T_H is the Hawking temperature.

Next we calculate the first-order correction of the Schwarzschild black hole spacetimes. We first expand the wave function to the first order in $1/\omega^{(d-3)/(d-2)}$ as

$$\Psi = \Psi^{(0)} + \frac{1}{\omega^{(d-3)/(d-2)}} \Psi^{(1)}.$$

Then one can rewrite eq. (2.4) as

$$\mathcal{H}_0 \Psi^{(1)} + \mathcal{H}_1 \Psi^{(0)} = 0. \tag{2.6}$$

The general solution of eq. (2.6) is

$$\Psi_\pm^{(1)} = \mathcal{C}\Psi_+^{(0)} \int_0^z \Psi_-^{(0)} \mathcal{H}_1 \Psi_\pm^{(0)} - \mathcal{C}\Psi_-^{(0)} \int_0^z \Psi_+^{(0)} \mathcal{H}_1 \Psi_\pm^{(0)}, \tag{2.7}$$

where $\mathcal{C} = -\frac{1}{\sin j\pi/2}$, and the wave function $\Psi_\pm^{(0)}$ are

$$\Psi_\pm^{(0)} = \sqrt{\frac{\pi z}{2}} J_{\pm j/2}(z).$$

Taking into consideration both the boundary conditions (2.2) and the behavior of wave function $F(z)$ as $z \to \pm\infty$ along the real axis, one may deduce

$$\mathcal{M}(r_H) \sim -\frac{\sin 3j\pi/2}{\sin j\pi/2}\left(1 + \frac{\xi_- - K}{\omega^{(d-3)/(d-2)}}\right),$$

leading to the generic expression of d-dimensional QN frequencies expressed as

$$\frac{\omega}{T_H} = (2n+1)\pi i + \ln(1 + 2\cos j\pi) + \frac{corr_d}{(n+1/2)^{(d-3)/(d-2)}},$$

where we have defined $\xi_\pm = c_{\pm\pm} e^{\pm ij\pi/2} - c_{+-}$, and

$$c_{\pm\pm} = \mathcal{C} \int_0^\infty \Psi_\pm^{(0)} \mathcal{H}_1 \Psi_\pm^{(0)}.$$

And K is defined as

$$K = e^{i\pi(3+6\eta)/4}\left[\xi_+ \sin(\frac{6\eta-3}{4}\pi) + i\xi_- \cos(\frac{6\eta-3}{4}\pi) - \xi \cos(\frac{6\eta-3}{4}\pi)\cot\frac{3j\pi}{2}\right],$$

where $\eta = (d-4)/2(d-2)$ and $\xi = \xi_+ + \xi_-{}^2$.

For $d = 4$, we obtain the same result as [22]

$$corr_4 = (1-i)\frac{3l(l+1)+1-j^2}{24\sqrt{2}(\pi)^{3/2}}\frac{\sin(2j\pi)}{\sin(3j\pi/2)}\Gamma^2(1/4)\Gamma(1/4+j/2)\Gamma(1/4-j/2).$$

The results above are shown to be dimension dependent and related closely to l and j. It is reasonable that these d-dimensional frequencies would indicate some information about back ground spacetime and perturbation types as shown in lowest QNMs. It approaches a constant for the real part of the QN frequency once we let $n \to \infty$ as expected in the literature [15, 16, 28]. In addition, [22] showed that the results do agree with the result from a WKB analysis [29], as well as the numerical results [4] for scalar perturbations and gravitational perturbations, respectively.

2.2 The Reissner-Nordström case

Now we discuss the QNMs of the Reissner-Nordström(RN) d-dimensional black hole including first-order corrections. The calculation for zeroth-order was first done in [16], and latter in [17]. In order to state more clearly, we start with the zeroth-order calculation. For Reissner-Nordström black hole, we have

$$f(r) = 1 - \frac{2m}{r^{d-3}} + \frac{q^2}{r^{2d-6}},$$

with the roots

$$r_n^\pm = \left|\left(m \pm \sqrt{m^2-q^2}\right)^{\frac{1}{d-3}}\right|\exp\left(\frac{2\pi i}{d-3}n\right), \quad n = 0, 1, \ldots, d-4,$$

where q is the charge of the black hole[3].

Again we define

$$F(r_*) = e^{i\omega r_*}\Psi(r_*).$$

Then the clockwise monodromy of $F(r_*)$ around the outer horizon $r = r_+$ can be obtained by continuing the coordinate r analytically into the complex plane, i.e.,

$$\mathcal{M}(r_+) = e^{\frac{2\pi\omega}{k_+}},$$

where $k_+ = \frac{1}{2}f'(r_+)$ is the surface gravity at the outer horizon.

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[2]Throughout this work we have the same definitions for $\alpha_\pm$, $c_\pm$, $\xi_\pm$, ξ, and $\mathcal{C}$.

[3]One can relate it with the charge Q by

$$Q^2 = \frac{(d-2)(d-3)q^2}{8\pi G_d}.$$

Near the black hole singularity ($r \sim 0$), the tortoise coordinate may be expanded as

$$r_* = \int \frac{dr}{f(r)} = \frac{1}{2d-5} \frac{r^{2d-5}}{q^2} + \frac{2m}{3d-8} \frac{r^{3d-8}}{q^4} + \cdots . \tag{2.8}$$

One can easily learn from (2.8) that $r_* \to \infty$ as $q \to 0$. In fact, in our procedure for expanding (2.8), we have assumed $\frac{r}{r_0^-} \ll 1$. As a result, instead of expanding potential to the first order in $1/\omega^{(d-3)/(2d-5)}$, we must expand it in $1/\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}$. After defining $z = \omega r_*$, the potential for the three different type perturbations can be then expanded, respectively, as (appendix A)

$$V[z] \sim -\frac{\omega^2}{4z^2} \left\{ 1 - j^2 - W(j) \left(\frac{z}{(r_0^-)^{2d-5}\omega} \right)^{(d-3)/(2d-5)} + \cdots \right\}, \tag{2.9}$$

where

$$W(j) = \begin{cases} W_{RNT} & j = j_T, \\ W_{RNV\pm} & j = j_{V\pm}, \\ W_{RNS\pm} & j = j_{S\pm}, \end{cases}$$

and the explicit expressions of W_{RNT}, $W_{RNV\pm}$ and $W_{RNS\pm}$ can be found in appendix A. Then the Schrödinger-like wave equation (2.1) with the potential (2.9) can be depicted as

$$\left(\mathcal{H}_0 + \left[(r_0^-)^{2d-5}\omega \right]^{-\frac{d-3}{2d-5}} \mathcal{H}_1 \right) \Psi = 0, \tag{2.10}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined as

$$\mathcal{H}_0 = \frac{d^2}{dz^2} + \left[\frac{1-j^2}{4z^2} + 1 \right], \quad \mathcal{H}_1 = -\frac{W(j)}{4} z^{-\frac{3d-7}{2d-5}}.$$

Taking into account of $\omega \to \infty$, the zeroth-order wave equation becomes

$$\mathcal{H}_0 \Psi^{(0)} = 0,$$

with general solutions in the form of eq. (2.5). According to the boundary conditions (2.2), one can define

$$F^{(0)}(z) = F_+^{(0)}(z) - e^{-\pi j i/2} F_-^{(0)}(z),$$

which approaches $-e^{-i\alpha_+} \sin \frac{j\pi}{2}$ as $z \to +\infty$. This holds at point A in figure [4] 1. By going around an arc of angle of $\frac{2\pi}{2d-5}$ in complex r-plane (rotating from A to the next branch), z rotates through an angle of 2π in z-plane, leading to the wave function
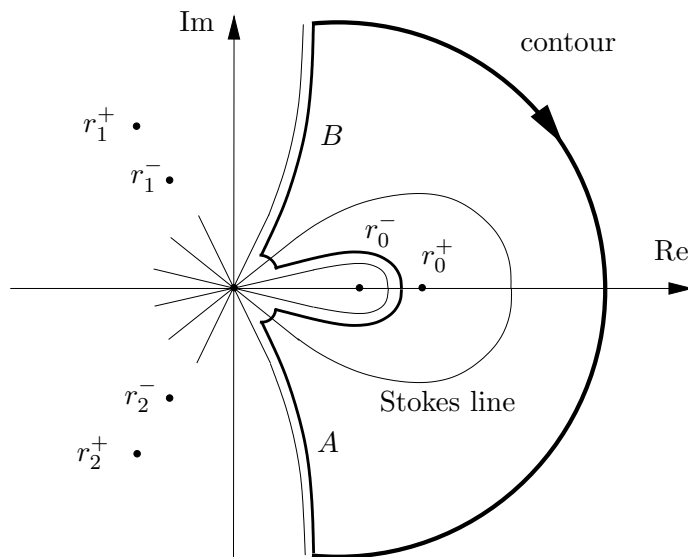
$$F^{(0)}(e^{2\pi i}z) = \frac{1}{2} \left[e^{2iz} \left(e^{3i\alpha_+} - e^{-ij\pi/2} e^{3i\alpha_-} \right) + \left(e^{5i\alpha_+} - e^{-ij\pi/2} e^{5i\alpha_-} \right) \right]. \tag{2.11}$$

As one follows the contour around the inner horizon $r = r_-$, the wave function will be of the form

$$F^{(0)}(z-\delta) = A_+ F_+^{(0)}(z-\delta) - A_- F_-^{(0)}(z-\delta),$$

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[4] All figures in this paper are provided by J. Natário and R. Schiappa in [17].

Figure 1: Stokes line for the Reissner-Nordström black hole, along with the chosen contour for monodromy matching, in the case $d = 6$ (we refer the reader to [17] for detail, and a more complete list of figures in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$).

where

$$\delta = \frac{2\omega\pi i}{f'(r_-)} = \frac{\omega\pi i}{k_-},$$

and $k_- = \frac{1}{2}f'(r_-)$ is the surface gravity at the inner horizon. Notice that $z - \delta < 0$ on this branch, one can easily obtain

$$F^{(0)}(z - \delta) = \frac{1}{2}\left[e^{2i(z-\delta)}\left(A_+ e^{i\alpha_+} - A_- e^{i\alpha_-}\right) + \left(A_+ e^{-i\alpha_+} - A_- e^{-i\alpha_-}\right)\right], \qquad (2.12)$$

where we have used

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z + \frac{\nu\pi}{2} + \frac{\pi}{4}\right), \qquad z \ll -1.$$

Since on the same branch, we must let $F^{(0)}(z - \delta) = F^{(0)}(e^{2\pi i}z)$, and hence it is easily seen from eqs. (2.11) and (2.12) that we must have

$$e^{3i\alpha_+} - e^{-ij\pi/2}e^{3i\alpha_-} = A_+ e^{i\alpha_+} e^{-2i\delta} - A_- e^{i\alpha_-} e^{-2i\delta},$$

$$e^{5i\alpha_+} - e^{-ij\pi/2}e^{5i\alpha_-} = A_+ e^{-i\alpha_+} - A_- e^{-i\alpha_-}.$$

Then we obtain

$$A_+ = 2e^{2i\delta}e^{\frac{\pi i}{2}}\cos\frac{j\pi}{2} - e^{i\pi\frac{3-j}{2}}\frac{\sin(3j\pi/2)}{\sin(j\pi/2)},$$

$$A_- = 2e^{2i\delta}e^{i\pi\frac{1-j}{2}}\cos\frac{j\pi}{2} - e^{\frac{3i\pi}{2}}\frac{\sin(3j\pi/2)}{\sin(j\pi/2)}. \qquad (2.13)$$

Finally, we must rotate to the branch containing point B. This makes $z - \delta$ rotate through an angle of 2π, leading to the wave function

$$F^{(0)}(e^{2\pi i}(z - \delta)) = 2e^{2i\delta}e^{-i\alpha_+}\cos(\frac{j\pi}{2})\sin(j\pi) + e^{-i\alpha_+}\sin(\frac{3j\pi}{2}).$$

Consequently, we deduce that

$$e^{\frac{2\pi\omega}{k_+}} = -2e^{2i\delta}[1 + \cos(j\pi)] - (1 + 2\cos(j\pi)), \qquad (2.14)$$

which agrees with the result shown in [17].

Next we compute the first-order correction of the asymptotic QNMs of the d-dimensional RN black hole using the monodromy method. The same reason as mentioned above, we must expand the wave function to the first order in $1/\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}$ as

$$\Psi = \Psi^{(0)} + \frac{1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\Psi^{(1)}.$$

Then one can rewrite eq. (2.10) as

$$\mathcal{H}_0\Psi^{(1)} + \mathcal{H}_1\Psi^{(0)} = 0.$$

The general solution of this equation, as mentioned in the last subsection, is

$$\Psi_{\pm}^{(1)} = \mathcal{C}\Psi_{+}^{(0)}\int_0^z \Psi_{-}^{(0)}\mathcal{H}_1\Psi_{\pm}^{(0)} - \mathcal{C}\Psi_{-}^{(0)}\int_0^z \Psi_{+}^{(0)}\mathcal{H}_1\Psi_{\pm}^{(0)}. \qquad (2.15)$$

The behavior as $z \gg 1$ is found to be

$$\Psi_{\pm}^{(1)}(z) \sim c_{-\pm}\cos(z - \alpha_+) - c_{+\pm}\cos(z - \alpha_-).$$

After defining

$$\Psi = \Psi_{+}^{(0)} + \frac{1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\Psi_{+}^{(1)} -$$
$$-e^{-ij\pi/2}\left(1 - \frac{\xi}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right)\left(\Psi_{-}^{(0)} + \frac{1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\Psi_{-}^{(1)}\right),$$

and using boundary condition (2.2), one finds that the wave function $F(z)$ approaches

$$F(z) \sim -e^{-i\alpha_+}\sin\frac{j\pi}{2}\left[1 - \frac{\xi_-}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right],$$

as $z \to \infty$.

The function $\Psi_{\pm}^{(1)}$ defined in (2.15) follows that

$$\Psi_{\pm}^{(1)} = z^{1\pm j/2+\eta}G_{\pm}(z),$$

where $\eta = -1/(4d - 10)$, and $G_{\pm}(z)$ are even analytic functions of z. By going around an arc of angle of $\frac{2\pi}{2d-5}$ in complex r-plane (rotating from A to the next branch), z rotates through an angle of 2π in z-plane, leading to the wave function

$$F(e^{2\pi i}z) = F^{(0)}(e^{2\pi i}z) + \frac{e^{i\pi(2+j+2\eta)}}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}$$
$$\times\left[F_{+}^{(1)}(z) - e^{-5j\pi i/2}\left(F_{-}^{(1)}(z) - \xi e^{-i\pi(1+2\eta)}F_{-}^{(0)}(z)\right)\right].$$

As one follows the contour around the inner horizon $r = r_-$, the wave function will be of the form

$$F(z - \delta) = \tilde{A}_+ F_+(z - \delta) - \tilde{A}_- F_-(z - \delta),$$

where again

$$\delta = \frac{2\omega\pi i}{f'(r_-)} = \frac{\omega\pi i}{k^-},$$

and $k^- = \frac{1}{2}f'(r_-)$ is the surface gravity at the inner horizon. For highly damped QNMs ($\omega \gg 1$), approximately, we have

$$\tilde{A}_+ = A_+, \quad \tilde{A}_- = A_-.$$

Finally, we must rotate to the branch containing point B. This makes $z - \delta$ rotate through an angle of 2π, leading to the wave function

$$F(e^{2\pi i}(z - \delta)) = e^{2i(z-\delta)}B + \frac{1}{2}\left(A_+ e^{3i\alpha_+} - A_- e^{3i\alpha_-}\right)$$
$$+ \frac{e^{2\eta\pi i}}{2\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\left[A_+ \xi_+ e^{3i\alpha_+} + A_- \xi_- e^{3i\alpha_-} + A_- \xi e^{-2\eta\pi i}e^{3i\alpha_-}\right],$$

where B is a constant which is not needed in our calculation because $e^{2i(z-\delta)} \to 0$ as $z \to -\infty$. Consequently, as one uses the explicit expression of A_+ and A_- in eq. (2.13), we deduce that

$$F(e^{2\pi i}(z - \delta)) = 2e^{2i\delta}e^{-i\alpha}\cos\left(\frac{j\pi}{2}\right)\sin(j\pi)\left(1 - \frac{K_1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right)$$
$$+ e^{-i\alpha}\sin\left(\frac{3j\pi}{2}\right)\left(1 - \frac{K_2}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right),$$

where

$$K_1 = e^{i\pi(\eta+3/2)}[\xi_+ \sin(\eta\pi) - \xi \cos(\eta\pi)\cot(j\pi) + i\xi_- \cos(\eta\pi)],$$
$$K_2 = e^{i\pi(\eta+3/2)}[\xi_+ \sin(\eta\pi) - \xi \cos(\eta\pi)\cot(j\pi/2) + i\xi_- \cos(\eta\pi)].$$

Finally, we have

$$e^{\frac{2\pi\omega}{k_+}} = -2e^{2i\delta}[1 + \cos(j\pi)]\left(1 + \frac{\xi_- - K_1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right)$$
$$- [1 + 2\cos(j\pi)]\left(1 + \frac{\xi_- - K_2}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right),$$

leading to the generic expression of d-dimensional QN frequencies

$$e^{\frac{\omega}{T_H^+}} + 2e^{-\frac{\omega}{T_H^-}}[1 + \cos(j\pi)]\left(1 + \frac{\xi_- - K_1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right)$$
$$+ [1 + 2\cos(j\pi)]\left(1 + \frac{\xi_- - K_2}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\right) = 0,$$

where T_H^+ and T_H^- represent the Hawking temperature at the outer and inner horizons (notice that $T_H^- < 0$).

For $d = 4$, $j_T = j_{S\pm} = 1/3$, $j_{V\pm} = 5/3$, one obtains

$$e^{\frac{\omega}{T_H^+}} + 3e^{-\frac{\omega}{T_H^-}}\left(1 - \frac{(\sqrt{3} - i)\mathcal{E}_1}{[(r_0^-)^3\omega]^{1/3}}\right) + 2\left(1 - \frac{(\sqrt{3} - i)\mathcal{E}_2}{[(r_0^-)^3\omega]^{1/3}}\right) = 0,$$

where

$$\mathcal{E}_1 = \frac{1 + \sqrt{3}i}{4}[\xi_+ - \xi_- + \sqrt{3}\cot(j\pi)], \quad \mathcal{E}_2 = \frac{1 + \sqrt{3}i}{4}[\xi_+ - \xi_- + \sqrt{3}\cot(j\pi/2)].$$

In the case of $q \ll m$, one has $-\frac{1}{T_H^-} \to 0$, which leads to

$$\frac{\omega}{T_H^+} \sim (2n + 1)\pi i + \log 5 - \frac{(\sqrt{3} - i)(\mathcal{E}_1 + \mathcal{E}_2)}{5r_0^-[(8n + 4)k_+]^{1/3}}.$$

In the case of $q \to m$, one has $\frac{1}{T_H^+} \approx -\frac{1}{T_H^-}$, which leads to

$$\frac{\omega}{T_H^+} \sim (2n + 1)\pi i - \log 2 + \frac{(\sqrt{3} - i)(\mathcal{E}_1 - 4\mathcal{E}_2)}{4r_0^-[(8n + 4)k_+]^{1/3}}.$$

In other cases, approximately, one has

$$e^{\frac{\omega}{T_H^+}} + 3e^{-\frac{\omega}{T_H^-}}\left(1 - \frac{(\sqrt{3} - i)\mathcal{E}_1}{r_0^-[(8n + 4)k_+]^{1/3}}\right) + 2\left(1 - \frac{(\sqrt{3} - i)\mathcal{E}_2}{r_0^-[(8n + 4)k_+]^{1/3}}\right) = 0, \quad (2.16)$$

In order to obtain an explicit expression of $\mathcal{E}_1$ and $\mathcal{E}_2$, we need the integral

$$\mathcal{J}(\nu, \mu) \equiv \int_0^\infty dz\, z^{-2/3} J_\nu(z) J_\mu(z) = \frac{\Gamma(\frac{2}{3})\Gamma(\frac{\mu + \nu + 1/3}{2})}{\sqrt[3]{4}\,\Gamma(\frac{\nu - \mu + 5/3}{2})\Gamma(\frac{\mu + \nu + 5/3}{2})\Gamma(\frac{\mu - \nu + 5/3}{2})}.$$

As a result, we obtain

$$\xi_+ = \frac{\pi W(j)}{8\sin(j\pi/2)}\left[\mathcal{J}(j/2, j/2)e^{ij\pi/2} - \mathcal{J}(j/2, -j/2)\right],$$

$$\xi_- = \frac{\pi W(j)}{8\sin(j\pi/2)}\left[\mathcal{J}(-j/2, -j/2)e^{-ij\pi/2} - \mathcal{J}(j/2, -j/2)\right],$$

Immediately, one has

$$\mathcal{E}_1 = -\frac{(\sqrt{3} - i)W(j)}{64\sqrt[3]{4}\pi^2} \times \frac{\cos(3j\pi/2)}{\cos^2(j\pi/2)} \cdot \Gamma(2/3)\Gamma(1/6 + j/2)\Gamma(1/6 - j/2)\Gamma^2(1/6),$$

$$\mathcal{E}_2 = -\frac{(\sqrt{3} - i)W(j)}{32\sqrt[3]{4}\pi^2} \cdot \Gamma(2/3)\Gamma(1/6 + j/2)\Gamma(1/6 - j/2)\Gamma^2(1/6).$$

In this way, one can easily obtain the value of QN frequencies in eq. (2.16). For tensor type perturbations, $W_{RNT} = 0$, so $\mathcal{E}_1 = \mathcal{E}_2 = 0$, leading to the same results as the zeroth-order asymptotic QN frequencies. However, it seems unavailable for scalar

type perturbations, since in this case we have $j \to 1/3$, which may induce the integral $\mathcal{J}(-1/3, -1/3)$ approaches infinity. It is interesting to investigate this problem in detail. Is there any other methods can avoid this singularity? For vector type perturbations, we have $j \to 5/3$, which lead to

$$
\begin{aligned}
\mathcal{E}_1 &\sim 0, \\
\mathcal{E}_2 &\sim (0.3734 - 0.2156i) \left(1 \pm \sqrt{9 + 4\ell(\ell+1)q^2 - 8q^2}\right)(1 - \sqrt{1-q^2})q^{-4/3},
\end{aligned}
$$

with $m = 1$. From here we find: (1) in order to insure $r_0^- [(8n+4)k_+]^{1/3} \gg 1$ as one calculates the QN frequencies of the first-order correction, the imaginary part of the frequencies (or the modes n) needs bigger values for a black hole with small charge, since $r_0^- \sim q^2 \to 0$ as $q \to 0$. This confirms the prediction made by Neitzke in [30]: the required n diverges as $q \to 0$, and the corrections would blow up this divergence; (2) just like the case of zeroth-order QN frequencies, the $q \to 0$ limit of RN $corr_4$ does not yield Schwarzschild $corr_4$, and the same thing happens in the limit of $q^2 \to m^2$, as we shall see in the following; (3) the first-order correction to the asymptotic QN frequencies are shown to be dimension dependent and related closely to ℓ, j, and the charge q.

QN frequencies of RN black holes were calculated numerically by Berti and Kokkotas in [31]. They found that very highly damped QNMs of RN black holes have an oscillatory behavior as a function of the charge. Their results have a good agreement with the zeroth-order analytical formula (2.14). However, it is necessary to perform further checks to our first-order corrected results both in four and higher dimensional black hole spacetime.

2.3 The extremal Reissner-Nordström case

Now we discuss the QNMs of the extremal Reissner-Nordström d-dimensional black hole including first-order corrections. The calculation for zeroth-order was done in [17], where they found that the $q^2 \to m^2$ limit of RN QN frequencies does not yield extremal RN QN frequencies. In this case, the outer horizon approaches the inner horizon.

Again we define

$$
F(r_*) = e^{i\omega r_*} \Psi(r_*).
$$

Then the clockwise monodromy of $F(r_*)$ around the horizon $r = r_0$ can be obtained by continuing the coordinate r analytically into the complex plane, i.e.,
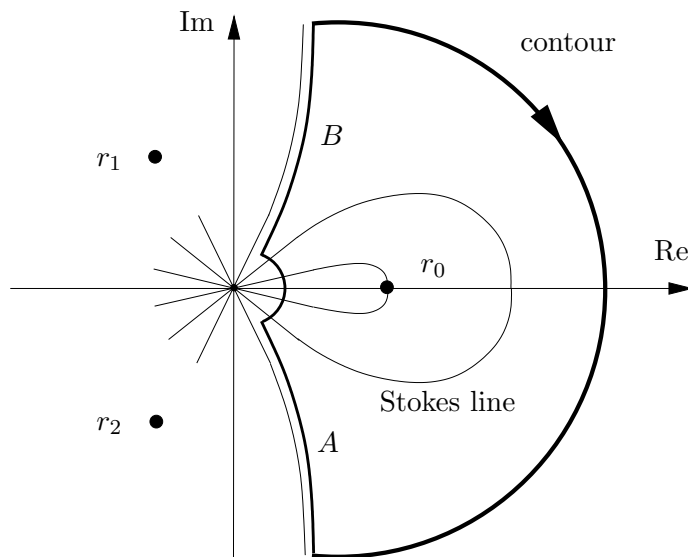
$$
\mathcal{M}(r_0) = e^{\frac{2\pi\omega}{k_0}},
$$

where $k_0 = \frac{(d-3)^2}{2(d-2)m^{1/d}}$.

Near the black hole singularity ($r \sim 0$), the tortoise coordinate may be expanded as

$$
r_* = \int \frac{dr}{f(r)} = \frac{1}{2d-5} \frac{r^{2d-5}}{m^2} + \frac{2}{3d-8} \frac{r^{3d-8}}{m^3} + \cdots .
$$

Again, when we define $z = \omega r_*$, the potential for the three different type perturbations can be expanded, respectively, as (appendix A)

$$
V[z] \sim -\frac{\omega^2}{4z^2} \left\{ 1 - j^2 - W(j)\left(\frac{z}{\omega}\right)^{(d-3)/(2d-5)} + \cdots \right\}, \tag{2.17}
$$

Figure 2: Stokes line for the extremal Reissner-Nordström black hole, along with the chosen contour for monodromy matching, in the case $d = 6$ (we refer the reader to [17] for detail, and a more complete list of figures in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$).

where

$$W(j) = \begin{cases} W_{RNT}^{ex} & j = j_T, \\ W_{RNV\pm}^{ex} & j = j_{V\pm}, \\ W_{RNS\pm}^{ex} & j = j_{S\pm}, \end{cases}$$

and the explicit expressions of W_{RNT}^{ex}, $W_{RNV\pm}^{ex}$ and $W_{RNS\pm}^{ex}$ can be found in appendix A. Then the Schrödinger-like wave equation (2.1) with the potential (2.17) can be depicted as

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{2d-5}} \mathcal{H}_1 \right) \Psi = 0, \tag{2.18}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined as

$$\mathcal{H}_0 = \frac{d^2}{dz^2} + \left[\frac{1 - j^2}{4z^2} + 1 \right], \quad \mathcal{H}_1 = -\frac{W(j)}{4} z^{-\frac{3d-7}{2d-5}}.$$

Taking into account of $\omega \to \infty$, the zeroth-order wave equation becomes

$$\mathcal{H}_0 \Psi^{(0)} = 0,$$

with general solutions in the form of eq. (2.5). According to the boundary conditions (2.2), one can define

$$F^{(0)}(z) = F_+^{(0)}(z) - e^{-\pi j i/2} F_-^{(0)}(z),$$

which approaches $-e^{-i\alpha_+} \sin \frac{j\pi}{2}$ as $z \to +\infty$. This holds at point A in figure 2. By going around an arc of angle of $\frac{5\pi}{2d-5}$ in complex r-plane (rotating from A to B), z rotates through an angle of 5π in z-plane, leading to the wave function

$$F^{(0)}(e^{5\pi i} z) = -e^{-i\alpha_+} \sin \frac{5j\pi}{2}.$$

Consequently, we deduce that

$$e^{\frac{2\pi\omega}{k}} = \frac{\sin\frac{5j\pi}{2}}{\sin\frac{j\pi}{2}},$$

leading to the generic expression of d-dimensional QN frequencies

$$\frac{\omega}{T} = 2n\pi i + \ln(1 + 2\cos j\pi + 2\cos 2j\pi),$$

where $T = 2\pi k_0$.

Next, we calculate the first-order correction to the asymptotic frequencies. Again we expand the wave function to the first order in $1/\omega^{(d-3)/(2d-5)}$ as

$$\Psi = \Psi^{(0)} + \frac{1}{\omega^{(d-3)/(2d-5)}}\Psi^{(1)}.$$

Then one can rewrite eq. (2.18) as

$$\mathcal{H}_0\Psi^{(1)} + \mathcal{H}_1\Psi^{(0)} = 0.$$

The general solution of this equation, as mentioned in the last subsection, is

$$\Psi^{(1)}_\pm = \mathcal{C}\Psi^{(0)}_+ \int_0^z \Psi^{(0)}_- \mathcal{H}_1\Psi^{(0)}_\pm - \mathcal{C}\Psi^{(0)}_- \int_0^z \Psi^{(0)}_+ \mathcal{H}_1\Psi^{(0)}_\pm. \tag{2.19}$$

The behavior as $z \gg 1$ is found to be

$$\Psi^{(1)}_\pm(z) \sim c_{-\pm}\cos(z - \alpha_+) - c_{+\pm}\cos(z - \alpha_-).$$

After defining

$$\Psi = \Psi^{(0)}_+ + \frac{1}{\omega^{(d-3)/(2d-5)}}\Psi^{(1)}_+ -$$
$$- e^{-ij\pi/2}\left(1 - \frac{\xi}{\omega^{(d-3)/(2d-5)}}\right)\left(\Psi^{(0)}_- + \frac{1}{\omega^{(d-3)/(2d-5)}}\Psi^{(1)}_-\right),$$

and using boundary condition (2.2), one finds that the wave function $F(z)$ approaches

$$F(z) \sim -e^{-i\alpha_+}\sin\frac{j\pi}{2}\left[1 - \frac{\xi_-}{\omega^{(d-3)/(2d-5)}}\right],$$

as $z \to \infty$.

The function $\Psi^{(1)}_\pm$ defined in (2.19) follows that

$$\Psi^{(1)}_\pm = z^{1\pm j/2+\eta}G_\pm(z),$$

where $\eta = -1/(4d - 10)$, and $G_\pm(z)$ are even analytic functions of z. By going around an arc of angle of $\frac{5\pi}{2d-5}$ in complex r-plane, z rotates through an angle of 5π in z-plane, leading to the wave function

$$F(e^{5\pi i}z) = e^{-2iz}B + \frac{1}{2}\left(e^{9i\alpha_+} - e^{-ij\pi/2}e^{9i\alpha_-}\right)$$
$$+ \frac{e^{5\eta\pi i}}{2\omega^{(d-3)/(2d-5)}}\left[-i\xi_+ e^{9i\alpha_+} - i\xi_- e^{-ij\pi/2}e^{9i\alpha_-} + \xi e^{-5\eta\pi i}e^{-ij\pi/2}e^{9i\alpha_-}\right],$$

where B is a constant which is not needed in our calculation because $e^{-2iz} \to 0$ as $z \to +\infty$. Finally, we have

$$e^{\frac{2\pi\omega}{k}} = [1 + 2\cos(j\pi) + 2\cos(2j\pi)] \left(1 + \frac{\xi_- - K}{\omega^{(d-3)/(2d-5)}}\right),$$

where

$$K = e^{(17+10\eta)\pi i/4} \left[\xi_+ \sin\left(\frac{3+10\eta}{4}\pi\right) - \xi \cos\left(\frac{3+10\eta}{4}\pi\right) \cot\left(\frac{5j\pi}{2}\right) \right.$$
$$\left. + i\xi_- \cos\left(\frac{3+10\eta}{4}\pi\right)\right],$$

leading to the generic expression of d-dimensional QN frequencies

$$\frac{\omega}{T} = 2n\pi i + \ln\left[1 + 2\cos(j\pi) + 2\cos(2j\pi)\right] + \frac{corr_d}{n^{(d-3)/(2d-5)}}.$$

For $d = 4$, one obtains

$$corr_4 = \frac{\sqrt{3}i - 1}{4\sqrt[3]{k_0}} \left[\sqrt{3}\xi_+ - \sqrt{3}\xi_- - \xi \cot(5j\pi/2)\right]. \tag{2.20}$$

In order to obtain an explicit expression of $corr_4$, we need the integral

$$\mathcal{J}(\nu,\mu) \equiv \int_0^\infty dz z^{-2/3} J_\nu(z) J_\mu(z) = \frac{\Gamma(\frac{2}{3})\Gamma(\frac{\mu+\nu+1/3}{2})}{\sqrt[3]{4}\Gamma(\frac{\nu-\mu+5/3}{2})\Gamma(\frac{\mu+\nu+5/3}{2})\Gamma(\frac{\mu-\nu+5/3}{2})}.$$

As a result, we obtain

$$\xi_+ = \frac{\pi W(j)}{8\sin(j\pi/2)} \left[\mathcal{J}(j/2, j/2)e^{ij\pi/2} - \mathcal{J}(j/2, -j/2)\right],$$
$$\xi_- = \frac{\pi W(j)}{8\sin(j\pi/2)} \left[\mathcal{J}(-j/2, -j/2)e^{-ij\pi/2} - \mathcal{J}(j/2, -j/2)\right],$$

Immediately, one has

$$corr_4 = -\frac{W(j)}{8\sqrt[3]{4k_0}\pi^2} \times \frac{\cos\frac{5j\pi}{2} - \cos\frac{7j\pi}{2} - (1-\sqrt{3}i)\cos\frac{3j\pi}{2}}{\sin\frac{j\pi}{2}\sin\frac{5j\pi}{2}}$$
$$\cdot \Gamma(2/3)\Gamma(1/6 + j/2)\Gamma(1/6 - j/2)\Gamma^2(1/6).$$

In this way, one can easily obtain the explicit expression of $corr_4$ of eq. (2.20). For tensor and scalar type perturbations, $W_{RNT}^{ex} = W_{RNS\pm}^{ex} = 0$, so $corr_4 = 0$, leading to the same results as the zeroth-order asymptotic QN frequencies. For vector type perturbations, we have $j \to 5/3$. Strangely, this also leads to $corr_4 \to 0$. It is interesting to investigate whether the first-order corrections for any $d > 3$ extremal RN black holes have a same behavior — they all approach zero, independent of the dimension. Another point is that in the limit of $q^2 \to m^2$, $corr_4$ of RN black hole does not approach zero — they have different correction terms.

2.4 The Schwarzschild de Sitter case

Now we compute the quasinormal modes of the Schwarzschild dS d-dimensional black hole including first-order corrections. We start with the zeroth-order calculation. For Schwarzschild de Sitter(SdS) black hole, we have

$$f(r) = 1 - \frac{2m}{r^{d-3}} - \lambda r^2,$$

with the roots

$$r_n = r_H, r_C, r_1, r_1^*, \ldots, r_{\frac{d-4}{2}}, r_{\frac{d-4}{2}}^*, \tilde{r},$$

where $\lambda > 0$ is the black hole background parameter related to the cosmological constant Λ by

$$\Lambda = \frac{1}{2}(d-1)(d-2)\lambda,$$

and r_n^* represents the conjugate of r_n. Here we have defined

$$\tilde{r} = - \left(r_H + r_C + \sum_{i=1}^{(d-4)/2} (r_i + r_i^*) \right).$$

The clockwise monodromy of $\Psi(r_*)$ around the event horizon $r = r_H$ and the cosmological horizon $r = r_C$ can be obtained by continuing the coordinate r analytically into the complex plane, respectively

$$\mathcal{M}(r_H) = e^{\frac{\pi \omega}{k_H}},$$
$$\mathcal{M}(r_C) = e^{\frac{-\pi \omega}{k_C}},$$

where $k_H = \frac{1}{2}f'(r_H)$ and $k_C = \frac{1}{2}f'(r_C)$ are the surface gravity at the event horizon and the cosmological horizon, respectively.

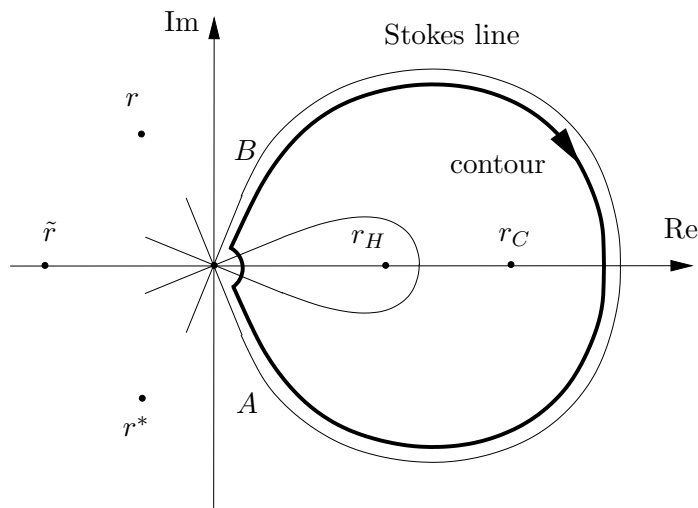Near the black hole singularity ($r \sim 0$), the tortoise coordinate may be expanded as

$$r_* = \int \frac{dr}{f(r)} = -\frac{1}{d-2}\frac{r^{d-2}}{2m} - \frac{1}{2d-5}\frac{r^{2d-5}}{(2m)^2} + \cdots.$$

After defining $z = \omega r_*$, the potential for the three different type perturbations can be expanded, respectively, as (appendix A)

$$V[z] \sim -\frac{\omega^2}{4z^2} \left\{ 1 - j^2 - W(j)\left(\frac{z}{\omega}\right)^{(d-3)/(d-2)} + \cdots \right\}, \tag{2.21}$$

where

$$W(j) = \begin{cases} W_{SdST} & j = j_T, \\ W_{SdSV} & j = j_V, \\ W_{SdSS} & j = j_S, \end{cases}$$

Figure 3: Stokes line for the Schwarzschild de Sitter black hole, along with the chosen contour for monodromy matching, in the case of dimension $d = 6$ (we refer the reader to [17] for detail, and a more complete list of figures in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$).

and the explicit expressions of W_{SdST}, W_{SdSV} and W_{SdSS} can be found in appendix A. Then the Schrödinger-like wave equation (2.1) with the potential (2.21) can be depicted as

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}} \mathcal{H}_1 \right) \Psi = 0, \tag{2.22}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined as

$$\mathcal{H}_0 = \frac{d^2}{dz^2} + \left[\frac{1 - j^2}{4z^2} + 1 \right], \quad \mathcal{H}_1 = -\frac{W(j)}{4} z^{-\frac{d-1}{d-2}}.$$

Obviously, the zeroth-order wave equation can be written as

$$\mathcal{H}_0 \Psi^{(0)} = 0,$$

with general solutions in the form of

$$\Psi^{(0)}(z) = A_+ \sqrt{\frac{\pi z}{2}} J_{j/2}(z) + A_- \sqrt{\frac{\pi z}{2}} J_{-j/2}(z).$$

As one lets $z \to +\infty$, the wave function approaches

$$\Psi^{(0)}(z) \sim \left(A_+ e^{-i\alpha_+} + A_- e^{-i\alpha_-} \right) \frac{e^{iz}}{2} + \left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-} \right) \frac{e^{-iz}}{2}.$$

This holds at point A in figure 3. By going around an arc of angle of $\frac{3\pi}{d-2}$ in complex r-plane (rotating from A to B), z rotates through an angle of 3π in z-plane, leading to the wave function

$$\Psi^{(0)}(z) \sim \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-} \right) \frac{e^{iz}}{2} + \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-} \right) \frac{e^{-iz}}{2}.$$

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Consequently, we have

$$\frac{A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}}{A_+ e^{-i\alpha_+} + A_- e^{-i\alpha_-}} e^{\frac{\pi\omega}{k_H} + \frac{\pi\omega}{k_C}} = \mathcal{M}(r_H)\mathcal{M}(r_C), \qquad (2.23)$$

$$\frac{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}} e^{-\frac{\pi\omega}{k_H} - \frac{\pi\omega}{k_C}} = \mathcal{M}(r_H)\mathcal{M}(r_C). \qquad (2.24)$$

Taking the condition for these equations to have nontrivial solutions (A_+, A_-) into account, one may easily obtain the final results as

$$\cosh\left(\frac{\pi\omega}{k_C} - \frac{\pi\omega}{k_H}\right) + (1 + 2\cos j\pi)\cosh\left(\frac{\pi\omega}{k_H} + \frac{\pi\omega}{k_C}\right) = 0. \qquad (2.25)$$

As discussed in ref. [17], the Stokes lines for $d = 4$ and $d = 5$ are a bit different from the case discussed in the previous calculation for $d = 6$. On account of the behavior of $r \sim \infty$, they found that the final result for $d = 4$ is the same as eq. (2.25). However, the result for $d = 5$ is changed. For $r \sim \infty$, the coefficient of e^{-iz} in the formula of the wave function keeps unchanged, while the coefficient of e^{iz} in there reverses sign as one rotates from the branch containing point B to the branch containing point A in the contour. Therefore one can complete this calculation by reversing the sign of eq. (2.23). In the end, we obtain

$$\sinh\left(\frac{\pi\omega}{k_C} - \frac{\pi\omega}{k_H}\right) + (1 + 2\cos j\pi)\sinh\left(\frac{\pi\omega}{k_H} + \frac{\pi\omega}{k_C}\right) = 0.$$

Next, we calculate the first-order correction to the asymptotic frequencies. Again we expand the wave function to the first order in $1/\omega^{(d-3)/(d-2)}$ as

$$\Psi = \Psi^{(0)} + \frac{1}{\omega^{(d-3)/(d-2)}} \Psi^{(1)}.$$

Then one can rewrite eq. (2.22) as

$$\mathcal{H}_0 \Psi^{(1)} + \mathcal{H}_1 \Psi^{(0)} = 0.$$

The general solution of this equation is

$$\Psi_\pm^{(1)} = \mathcal{C}\Psi_+^{(0)} \int_0^z \Psi_-^{(0)} \mathcal{H}_1 \Psi_\pm^{(0)} - \mathcal{C}\Psi_-^{(0)} \int_0^z \Psi_+^{(0)} \mathcal{H}_1 \Psi_\pm^{(0)}. \qquad (2.26)$$

The behavior as $z \gg 1$, as mentioned above, is found to be

$$\Psi_\pm^{(1)}(z) \sim c_{-\pm}\cos(z - \alpha_+) - c_{+\pm}\cos(z - \alpha_-).$$

After defining

$$a_{\pm\pm} = c_{\pm\pm}\omega^{-\frac{d-3}{d-2}}, \qquad u_\pm = 1 \pm a_{+-},$$

one finds that the wave function $\Psi(z)$ approaches

$$\Psi(z) = \Psi^{(0)}(z) + \omega^{-\frac{d-3}{d-2}}\Psi^{(1)}(z)$$
$$\sim (A_+ u_+ + A_- a_{--})\cos(z - \alpha_+) + (A_- u_- - A_+ a_{++})\cos(z - \alpha_-)$$

$$= \left[(A_+ u_+ + A_- a_{--}) e^{-i\alpha_+} + (A_- u_- - A_+ a_{++}) e^{-i\alpha_-} \right] \frac{e^{iz}}{2} +$$

$$\left[(A_+ u_+ + A_- a_{--}) e^{i\alpha_+} + (A_- u_- - A_+ a_{++}) e^{i\alpha_-} \right] \frac{e^{-iz}}{2}$$

as $z \to \infty$.

The function $\Psi_\pm^{(1)}$ defined in (2.26) follows that

$$\Psi_\pm^{(1)} = z^{1/2 \pm j/2 + \eta} G_\pm(z),$$

where $\eta = (d-3)/(d-2)$, and $G_\pm(z)$ are even analytic functions of z. By going around an arc of angle of $\frac{3\pi}{d-2}$ in complex r-plane, z rotates through an angle of 3π in z-plane, leading to the wave function

$$\Psi(z) \sim \left[(A_+ v_+ + A_- a_{--} e^{-3ij\pi}) e^{7i\alpha_+} + (A_- v_- - A_+ a_{++} e^{3ij\pi}) e^{7i\alpha_-} \right] \frac{e^{3\eta\pi i} e^{iz}}{2} +$$

$$\left[(A_+ v_+ + A_- a_{--} e^{-3ij\pi}) e^{5i\alpha_+} + (A_- v_- - A_+ a_{++} e^{3ij\pi}) e^{5i\alpha_-} \right] \frac{e^{3\eta\pi i} e^{-iz}}{2}.$$

as $z \to -\infty$. Here we have defined $v_\pm = e^{-3\eta\pi i} \pm a_{+-}$. We hence have the similar formulae as shown in (2.23) and (2.24), i.e.,

$$\frac{(A_+ v_+ + A_- a_{--} e^{-3ij\pi}) e^{7i\alpha_+} + (A_- v_- - A_+ a_{++} e^{3ij\pi}) e^{7i\alpha_-}}{(A_+ u_+ + A_- a_{--}) e^{-i\alpha_+} + (A_- u_- - A_+ a_{++}) e^{-i\alpha_-}}$$
$$e^{\frac{\pi\omega}{k_H} + \frac{\pi\omega}{k_C}} = e^{-3\eta\pi i} \mathcal{M}(r_H) \mathcal{M}(r_C), \quad (2.27)$$

$$\frac{(A_+ v_+ + A_- a_{--} e^{-3ij\pi}) e^{5i\alpha_+} + (A_- v_- - A_+ a_{++} e^{3ij\pi}) e^{5i\alpha_-}}{(A_+ u_+ + A_- a_{--}) e^{i\alpha_+} + (A_- u_- - A_+ a_{++}) e^{i\alpha_-}}$$
$$e^{-\frac{\pi\omega}{k_H} - \frac{\pi\omega}{k_C}} = e^{-3\eta\pi i} \mathcal{M}(r_H) \mathcal{M}(r_C). \quad (2.28)$$

In this way, one can easily obtain a set of equations with regard to A_+ and A_-

$$\left(s_1 e^{7i\alpha_+} - s_2 e^{-\frac{2\pi\omega}{k_C}} e^{-i\alpha_+} \right) A_+ + \left(s_3 e^{7i\alpha_-} - s_4 e^{-\frac{2\pi\omega}{k_C}} e^{-i\alpha_-} \right) A_- = 0,$$

$$\left(s_5 e^{5i\alpha_+} - s_6 e^{\frac{2\pi\omega}{k_H}} e^{i\alpha_+} \right) A_+ + \left(s_7 e^{5i\alpha_-} - s_8 e^{\frac{2\pi\omega}{k_H}} e^{i\alpha_-} \right) A_- = 0,$$

where

$$s_1 = (v_+ - a_{++} e^{-\frac{ij\pi}{2}}) e^{3\eta\pi i}, \quad s_2 = (u_+ - a_{++} e^{\frac{ij\pi}{2}}),$$
$$s_3 = (v_- + a_{--} e^{\frac{ij\pi}{2}}) e^{3\eta\pi i}, \quad s_4 = (u_- + a_{--} e^{-\frac{ij\pi}{2}}),$$
$$s_5 = (v_+ - a_{++} e^{\frac{ij\pi}{2}}) e^{3\eta\pi i}, \quad s_6 = (u_+ - a_{++} e^{-\frac{ij\pi}{2}}),$$
$$s_7 = (v_- + a_{--} e^{-\frac{ij\pi}{2}}) e^{3\eta\pi i}, \quad s_8 = (u_- + a_{--} e^{\frac{ij\pi}{2}}).$$

The condition for these equations to have nontrivial solutions (A_+, A_-) is then

$$\begin{vmatrix} s_1 e^{7i\alpha_+} - s_2 e^{-\frac{2\pi\omega}{k_C}} e^{-i\alpha_+} & s_3 e^{7i\alpha_-} - s_4 e^{-\frac{2\pi\omega}{k_C}} e^{-i\alpha_-} \\ s_5 e^{5i\alpha_+} - s_6 e^{\frac{2\pi\omega}{k_H}} e^{i\alpha_+} & s_7 e^{5i\alpha_-} - s_8 e^{\frac{2\pi\omega}{k_H}} e^{i\alpha_-} \end{vmatrix} = 0.$$

After some algebra, we obtain

$$\mathcal{A}_1 \cosh\left(\frac{\pi\omega}{k_C} - \frac{\pi\omega}{k_H} + \Delta_1\right) + \mathcal{A}_2 \cosh\left(\frac{\pi\omega}{k_H} + \frac{\pi\omega}{k_C} + \Delta_2\right) = 0, \qquad (2.29)$$

where

$$\mathcal{A}_1 = \sqrt{\left(s_1 s_7 e^{\frac{ij\pi}{2}} - s_3 s_5 e^{-\frac{ij\pi}{2}}\right)\left(s_4 s_6 e^{\frac{ij\pi}{2}} - s_2 s_8 e^{-\frac{ij\pi}{2}}\right)},$$

$$\mathcal{A}_2 = \sqrt{\left(s_1 s_8 e^{\frac{3ij\pi}{2}} - s_3 s_6 e^{-\frac{3ij\pi}{2}}\right)\left(s_4 s_5 e^{\frac{3ij\pi}{2}} - s_2 s_7 e^{-\frac{3ij\pi}{2}}\right)},$$

$$\Delta_1 = \frac{1}{2}\left[\log\left(s_1 s_7 e^{\frac{ij\pi}{2}} - s_3 s_5 e^{-\frac{ij\pi}{2}}\right) - \log\left(s_4 s_6 e^{\frac{ij\pi}{2}} - s_2 s_8 e^{-\frac{ij\pi}{2}}\right)\right],$$

$$\Delta_2 = \frac{1}{2}\left[\log\left(s_1 s_8 e^{\frac{3ij\pi}{2}} - s_3 s_6 e^{-\frac{3ij\pi}{2}}\right) - \log\left(s_4 s_5 e^{\frac{3ij\pi}{2}} - s_2 s_7 e^{-\frac{3ij\pi}{2}}\right)\right].$$

For zeroth-order asymptotic QN frequencies, one has $a_{\pm\pm} = 0$, and hence $u_\pm = 1$ and $v_\pm = e^{-3\eta\pi i}$. As a result, formula (2.29) reduces to (2.25).

As mentioned formerly, the Stokes lines for $d = 4$ and $d = 5$ are a bit different from the case discussed in the previous calculation for $d = 6$. Resorting to the behavior of $r \sim \infty$, one finds that the final result for $d = 4$ is shown to be the same as eq. (2.29). However, the result for $d = 5$ is changed. For $r \sim \infty$, the coefficient of e^{-iz} in the formula of the wave function keeps unchanged, while the coefficient of e^{iz} in there reverses sign as one rotates from the branch containing point B to the branch containing point A in the contour. Therefore one can complete this calculation by reversing the sign of eq. (2.27). In the end, we obtain

$$\mathcal{A}_1 \sinh\left(\frac{\pi\omega}{k_C} - \frac{\pi\omega}{k_H} + \Delta_1\right) + \mathcal{A}_2 \sinh\left(\frac{\pi\omega}{k_H} + \frac{\pi\omega}{k_C} + \Delta_2\right) = 0.$$

For $d = 4$ SdS black holes, one has $r_C \gg r_H$ as $0 < \lambda \ll 1$, as a result, one has $k_H \gg k_C$. In this case, we can neglect k_C compared to k_H. Taking eq. (2.25) into account, one finds the asymptotic QN frequencies in this case is the solutions of this formula

$$e^{\frac{\pi\omega}{k_C}} + e^{-\frac{\pi\omega}{k_C}} = 0.$$

So, we have $\frac{\omega}{T_C} = (2n+1)\pi i$. However, as one lets λ approaches its extremal value, i.e., $\lambda \to 1/27$ (with $m = 1$), one has $k_H \approx k_C$. So, in this limit, we can obtain the asymptotic QN frequencies by solving this formula

$$e^{2\frac{\pi\omega}{k_C}} + e^{-2\frac{\pi\omega}{k_C}} = 0,$$

which in turn derives $\frac{\omega}{T_C} = \frac{1}{2}(2n+1)\pi i$, where T_C is the Hawking temperature at cosmological horizon. This reminds us to conjecture that the frequencies have values between these two extremal values, i.e.,

$$\frac{\omega}{T_C} \simeq \chi(2n+1)\pi i + \mathrm{Re}\,\omega, \qquad \frac{1}{2} < \chi < 1, \qquad (2.30)$$

where χ is a parameter closely related to the cosmological constant λ. In fact, it would be an interesting work to investigate the relationship between these two parameters, by analytical methods or numerical ones. Re ω is the real part of the frequency.

In order to obtain the explicit expressions of $\mathcal{A}_1$, $\mathcal{A}_2$, Δ_1 and Δ_2, we need the integral

$$\int_0^\infty dz \, z^{-1/2} J_\nu(z) J_\mu(z) = \frac{\sqrt{\pi/2}\,\Gamma(\frac{\mu+\nu+1/2}{2})}{\Gamma(\frac{\nu-\mu+3/2}{2})\Gamma(\frac{\mu+\nu+3/2}{2})\Gamma(\frac{\mu-\nu+3/2}{2})}.$$

As a result, we obtain

$$s_1 = 1 - 2iU(j)\left(\cos\frac{j\pi}{2} - \sin\frac{j\pi}{2}\right), \qquad s_2 = 1 - 2iU(j)\left(\cos\frac{j\pi}{2} - \sin\frac{j\pi}{2}\right),$$

$$s_3 = 1 - 2iU(j)\left(\cos\frac{j\pi}{2} + \sin\frac{j\pi}{2}\right), \qquad s_4 = 1 - 2iU(j)\left(\cos\frac{j\pi}{2} + \sin\frac{j\pi}{2}\right),$$

$$s_5 = 1 - 2U(j)\left(\cos\frac{j\pi}{2} - \sin\frac{j\pi}{2}\right), \qquad s_6 = 1 + 2U(j)\left(\cos\frac{j\pi}{2} - \sin\frac{j\pi}{2}\right),$$

$$s_7 = 1 - 2U(j)\left(\cos\frac{j\pi}{2} + \sin\frac{j\pi}{2}\right), \qquad s_8 = 1 + 2U(j)\left(\cos\frac{j\pi}{2} + \sin\frac{j\pi}{2}\right),$$

where

$$U(j) = \frac{W(j)}{32\pi^{3/2}\sqrt{(2n+1)\chi k_C}}\Gamma^2(1/4)\Gamma(1/4 + j/2)\Gamma(1/4 - j/2).$$

After some algebra, we obtain the explicit expressions of $\mathcal{A}_1$, $\mathcal{A}_2$, Δ_1 and Δ_2 in terms of $o(1/\sqrt{n})$

$$\mathcal{A}_1 \simeq 2i\sin\frac{j\pi}{2},$$

$$\mathcal{A}_2 \simeq 2i\sin\frac{3j\pi}{2}\left[1 - 2iU(j)\left(\cos j\pi/2 + \frac{\cos 3j\pi/2}{1 + 2\cos j\pi}\right)\right],$$

$$\Delta_1 \simeq 0,$$

$$\Delta_2 \simeq 2U(j)\left(\cos j\pi/2 + \frac{\cos 3j\pi/2}{1 + 2\cos j\pi}\right).$$

All type perturbations, including tensor and scalar type perturbations $(j \to 0^+)$, and vector type perturbation $(j \to 2)$ have a same behavior that $\mathcal{A}_1$ and $\mathcal{A}_2$ approach zero in the limit. This makes eq. (2.29) automatically satisfied. As shown in [17], we first regard j as nonzero, then take the limit as $j \to 0$. In the end, eq. (2.29) becomes

$$\cosh\left(\frac{\pi\omega}{k_C} - \frac{\pi\omega}{k_H}\right) + (3 - 8iU(j))\cosh\left(\frac{\pi\omega}{k_H} + \frac{\pi\omega}{k_C} + \frac{8U(j)}{3}\right) = 0. \qquad (2.31)$$

Inserting the expressions of $W(j)$ (see appendix A), we can easily obtain the explicit expression of $U(j)$. In this way, the first-order correction to asymptotic QN frequencies of SdS black holes can be obtained by evaluating eq. (2.31). The result show that the correction term is closely related to ℓ, λ (though the parameter χ and k_C), and of course, n.

Numerical calculation on this case for very highly damped overtone first appeared in [32] and later in [33]. An analytical formula for four dimensional case was deduced

in [34]. They found that the analytical results are in good agreement with the numerical results. However, it is interesting to perform further checks to our first-order corrected results both in four and higher dimensional black hole spacetimes.

2.5 The Reissner-Nordström de Sitter case

Now we compute the quasinormal modes of the RN dS d-dimensional black hole including first-order corrections. We start with the zeroth-order calculation. For RN de Sitter black hole, we have

$$f(r) = 1 - \frac{2m}{r^{d-3}} + \frac{q^2}{r^{2d-6}} - \lambda r^2,$$

with the roots

$$r_n = r_+, r_-, r_C, r_1, r_1^*, \ldots, r_{d-4}, r_{d-4}^*, \tilde{r},$$

where $\lambda(> 0)$ is the black hole background parameter related to the cosmological constant Λ by

$$\Lambda = \frac{1}{2}(d-1)(d-2)\lambda,$$

and r_n^* represents the conjugate of r_n. Here we have defined

$$\tilde{r} = - \left(r_+ + r_- + r_C + \sum_{i=1}^{d-4}(r_i + r_i^*) \right).$$

The clockwise monodromy of $\Psi(r_*)$ around the outer horizon $r = r_+$ and the cosmological horizon $r = r_C$ can be obtained by continuing the coordinate r analytically into the complex plane, respectively

$$\mathcal{M}(r_+) = e^{\frac{\pi\omega}{k_+}},$$
$$\mathcal{M}(r_C) = e^{\frac{-\pi\omega}{k_C}},$$

where $k_+ = \frac{1}{2}f'(r_+)$ and $k_C = \frac{1}{2}f'(r_C)$ are the surface gravity at the outer horizon and the cosmological horizon, respectively.

Near the black hole singularity ($r \sim 0$), the tortoise coordinate may be expanded as

$$r_* = \int \frac{dr}{f(r)} = \frac{1}{2d-5}\frac{r^{2d-5}}{q^2} + \frac{2m}{3d-8}\frac{r^{3d-8}}{q^4} + \cdots,$$

In this procedure, we have assumed $\frac{r}{r_0^-} \ll 1$, where $r_0^- = \left(m - \sqrt{m^2 - q^2} \right)^{\frac{1}{d-3}}$, represents the inner horizon of the RN black hole. Again we must expand the potential to the first order in $1/\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}$ instead of $1/\omega^{(d-3)/(2d-5)}$. After defining $z = \omega r_*$, the potential for the three different type perturbations can be expanded, respectively, as (appendix A)

$$V[z] \sim -\frac{\omega^2}{4z^2} \left\{ 1 - j^2 - W(j) \left(\frac{z}{(r_0^-)^{2d-5}\omega} \right)^{(d-3)/(2d-5)} + \cdots \right\}, \tag{2.32}$$

where

$$W(j) = \begin{cases} W_{RNdST} & j = j_T, \\ W_{RNdSV\pm} & j = j_{V\pm}, \\ W_{RNdSS\pm} & j = j_{S\pm}, \end{cases}$$

and the explicit expressions of W_{RNdST}, $W_{RNdSV\pm}$ and $W_{RNdSS\pm}$ can be found in appendix A. Then the Schrödinger-like wave equation (2.1) with the potential (2.32) can be depicted as

$$\left(\mathcal{H}_0 + \left[(r_0^-)^{2d-5}\omega \right]^{-\frac{d-3}{2d-5}} \mathcal{H}_1 \right) \Psi = 0, \tag{2.33}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined as

$$\mathcal{H}_0 = \frac{d^2}{dz^2} + \left[\frac{1-j^2}{4z^2} + 1 \right], \quad \mathcal{H}_1 = -\frac{W(j)}{4} z^{-\frac{3d-7}{2d-5}}.$$

Obviously, the zeroth-order wave equation can be written as

$$\mathcal{H}_0 \Psi^{(0)} = 0,$$

with general solutions in the form of

$$\Psi^{(0)}(z) = A_+ \sqrt{\frac{\pi z}{2}} J_{j/2}(z) + A_- \sqrt{\frac{\pi z}{2}} J_{-j/2}(z).$$

As one lets $z \to +\infty$, the wave function approaches

$$\Psi^{(0)}(z) \sim \left(A_+ e^{-i\alpha_+} + A_- e^{-i\alpha_-} \right) \frac{e^{iz}}{2} + \left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-} \right) \frac{e^{-iz}}{2}. \tag{2.34}$$

This holds at point A in figure 4. By going around an arc of angle of $\frac{2\pi}{2d-5}$ in complex r-plane (rotating from A to the next branch), z rotates through an angle of 2π in z-plane, leading to the wave function

$$\Psi^{(0)}(z) \sim \left(A_+ e^{3i\alpha_+} + A_- e^{3i\alpha_-} \right) \frac{e^{iz}}{2} + \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-} \right) \frac{e^{-iz}}{2}.$$

As one follows the contour around the inner horizon $r = r_-$, the wave function will be of the form

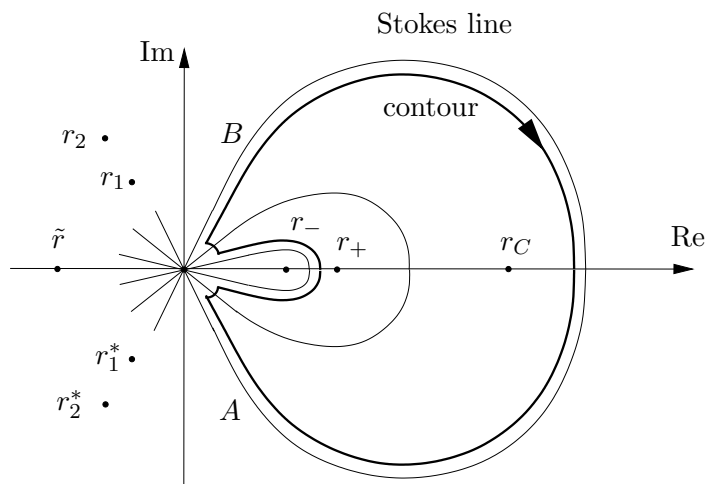$$\Psi^{(0)}(z - \delta) = B_+ \sqrt{\frac{\pi(z-\delta)}{2}} J_{j/2}(z-\delta) + B_- \sqrt{\frac{\pi(z-\delta)}{2}} J_{-j/2}(z-\delta),$$

where

$$\delta = \frac{2\omega\pi i}{f'(r_-)} = \frac{\omega\pi i}{k_-},$$

and $k_- = \frac{1}{2} f'(r_-)$ is the surface gravity at the inner horizon. Notice that $z - \delta < 0$ on this branch, one can easily obtain

$$\Psi^{(0)}(z - \delta) \sim \left(B_+ e^{i\alpha_+} + B_- e^{i\alpha_-} \right) \frac{e^{i(z-\delta)}}{2} + \left(B_+ e^{-i\alpha_+} + B_- e^{-i\alpha_-} \right) \frac{e^{-i(z-\delta)}}{2}, \tag{2.35}$$

Figure 4: Stokes line for the Reissner-Nordström de Sitter black hole, along with the chosen contour for monodromy matching, in the case of dimension $d = 6$ (we refer the reader to [17] for detail, and a more complete list of figures in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$).

where we have used

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z + \frac{\nu\pi}{2} + \frac{\pi}{4}\right), \qquad z \ll -1.$$

Since on the same branch, we must let $\Psi^{(0)}(z - \delta) = \Psi^{(0)}(e^{2\pi i}z)$, and hence it is easily seen from eqs. (2.34) and (2.35) that we have

$$A_+ e^{3i\alpha_+} + A_- e^{3i\alpha_-} = (B_+ e^{i\alpha_+} + B_- e^{i\alpha_-})e^{-i\delta}, \tag{2.36}$$

$$A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-} = (B_+ e^{-i\alpha_+} + B_- e^{-i\alpha_-})e^{i\delta}. \tag{2.37}$$

Finally, we should rotate to the branch containing point B. This makes $z - \delta$ rotate through an angle of 2π, leading to the wave function

$$\Psi^{(0)}(z - \delta) \sim \left(B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}\right)\frac{e^{i(z-\delta)}}{2} + \left(B_+ e^{3i\alpha_+} + B_- e^{3i\alpha_-}\right)\frac{e^{-i(z-\delta)}}{2}.$$

Consequently, we have

$$\frac{B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}}{A_+ e^{-i\alpha_+} + A_- e^{-i\alpha_-}} e^{-i\delta} e^{\frac{\pi\omega}{k_+} + \frac{\pi\omega}{k_C}} = \mathcal{M}(r_+)\mathcal{M}(r_C), \tag{2.38}$$

$$\frac{B_+ e^{3i\alpha_+} + B_- e^{3i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}} e^{i\delta} e^{-\frac{\pi\omega}{k_H} - \frac{\pi\omega}{k_C}} = \mathcal{M}(r_+)\mathcal{M}(r_C). \tag{2.39}$$

Taking the condition for these equations (2.36)–(2.39) to have nontrivial solutions (A_+, A_-, B_+, B_-) into account, one can easily obtain the final results

$$\cosh\left(\frac{\pi\omega}{k_+} - \frac{\pi\omega}{k_C}\right) + (1 + 2\cos(j\pi))\cosh\left(\frac{\pi\omega}{k_+} + \frac{\pi\omega}{k_C}\right)$$
$$+ 2(1 + \cos(j\pi))\cosh\left(\frac{\pi\omega}{k_+} + \frac{2\pi\omega}{k_-} + \frac{\pi\omega}{k_C}\right) = 0. \tag{2.40}$$

As discussed in ref [17], the Stokes lines for $d = 4$ and $d = 5$ are a bit different from the case discussed in the previous calculation for $d = 6$. On account of the behavior of $r \sim \infty$, they found that the final result for $d = 4$ is the same as eq. (2.40). However, the result for $d = 5$ is changed. For $r \sim \infty$, the coefficient of e^{-iz} in the formula of the wave function keeps unchanged, while the coefficient of e^{iz} in there reverses sign as one rotates from the branch containing point B to the branch containing point A in the contour. Therefore one can complete this calculation by reversing the sign of eq. (2.38). In the end, we obtain

$$\sinh\left(\frac{\pi\omega}{k_C} - \frac{\pi\omega}{k_+}\right) + (1 + 2\cos(2\pi/5))\sinh\left(\frac{\pi\omega}{k_+} + \frac{\pi\omega}{k_C}\right)$$
$$+ 2(1 + \cos(2\pi/5))\sinh\left(\frac{\pi\omega}{k_+} + \frac{2\pi\omega}{k_-} + \frac{\pi\omega}{k_C}\right) = 0.$$

Next, we calculate the first-order correction to the asymptotic frequencies. Again we expand the wave function to the first order in $1/\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}$ as

$$\Psi = \Psi^{(0)} + \frac{1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}}\Psi^{(1)}.$$

Then one can rewrite eq. (2.33) as

$$\mathcal{H}_0\Psi^{(1)} + \mathcal{H}_1\Psi^{(0)} = 0. \tag{2.41}$$

The general solution of this equation is

$$\Psi_\pm^{(1)} = \mathcal{C}\Psi_+^{(0)}\int_0^z \Psi_-^{(0)}\mathcal{H}_1\Psi_\pm^{(0)} - \mathcal{C}\Psi_-^{(0)}\int_0^z \Psi_+^{(0)}\mathcal{H}_1\Psi_\pm^{(0)}. \tag{2.42}$$

The behavior as $z \gg 1$, as mentioned above, is found to be

$$\Psi_\pm^{(1)}(z) \sim c_{-\pm}\cos(z - \alpha_+) - c_{+\pm}\cos(z - \alpha_-).$$

After defining

$$a_{\pm\pm} = c_{\pm\pm}\left[(r_0^-)^{2d-5}\omega\right]^{-\frac{d-3}{2d-5}}, \qquad u_\pm = 1 \pm a_{+-},$$

one finds that the wave function $\Psi(z)$ approaches

$$\Psi(z) = \Psi^{(0)}(z) + \left[(r_0^-)^{2d-5}\omega\right]^{-\frac{d-3}{2d-5}}\Psi^{(1)}(z)$$
$$\sim (A_+u_+ + A_-a_{--})\cos(z - \alpha_+) + (A_-u_- + A_+a_{++})\cos(z - \alpha_-)$$
$$= \left[(A_+u_+ + A_-a_{--})e^{-i\alpha_+} + (A_-u_- - A_+a_{++})e^{-i\alpha_-}\right]\frac{e^{iz}}{2} +$$
$$\left[(A_+u_+ + A_-a_{--})e^{i\alpha_+} + (A_-u_- - A_+a_{++})e^{i\alpha_-}\right]\frac{e^{-iz}}{2}$$

as $z \to \infty$.

The function $\Psi_\pm^{(1)}$ defined in (2.42) follows that

$$\Psi_\pm^{(1)} = z^{1\pm j/2+\eta}G_\pm(z),$$

where $\eta = -1/(4d-10)$, and $G_{\pm}(z)$ are even analytic functions of z. By going around an arc of angle of $\frac{2\pi}{2d-5}$ in complex r-plane (rotating from A to the next branch), z rotates through an angle of 2π in z-plane, leading to the wave function

$$\Psi(z) \sim \left[(A_+v_- - A_-a_{--}e^{-2ij\pi})e^{3i\alpha_+} + (A_-v_+ + A_+a_{++}e^{2ij\pi})e^{3i\alpha_-}\right]\frac{e^{2\eta\pi i}e^{iz}}{2} +$$
$$\left[(A_+v_- - A_-a_{--}e^{-2ij\pi})e^{5i\alpha_+} + (A_-v_+ + A_+a_{++}e^{2ij\pi})e^{5i\alpha_-}\right]\frac{e^{2\eta\pi i}e^{-iz}}{2}. \quad (2.43)$$

as $z \to -\infty$. Here we have defined $v_{\pm} = e^{-2\eta\pi i} \pm a_{+-}$.

As one follows the contour around the inner horizon $r = r_-$, the wave function will be of the form

$$\Psi(z-\delta) = \Psi^{(0)}(z-\delta) + \left[(r_0^-)^{2d-5}\omega\right]^{-\frac{d-3}{2d-5}}\Psi^{(1)}(z-\delta),$$

where

$$\delta = \frac{2\omega\pi i}{f'(r_-)} = \frac{\omega\pi i}{k_-},$$

and $k_- = \frac{1}{2}f'(r_-)$ is the surface gravity at the inner horizon. Notice that $z - \delta < 0$ on this branch. As one lets $z - \delta \to -\infty$, the wave function becomes

$$\Psi(z-\delta) \sim \left[(B_+u_+ + B_-a_{--})e^{i\alpha_+} + (B_-u_- - B_+a_{++})e^{i\alpha_-}\right]\frac{e^{i(z-\delta)}}{2} +$$
$$\left[(B_+u_+ + B_-a_{--})e^{-i\alpha_+} + (B_-u_- - B_+a_{++})e^{-i\alpha_-}\right]\frac{e^{-i(z-\delta)}}{2}, \quad (2.44)$$

where we used

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z + \frac{\nu\pi}{2} + \frac{\pi}{4}\right), \qquad z \ll -1.$$

Again, since on the same branch, we must let $\Psi(z-\delta) = \Psi(e^{2\pi i}z)$. Then it is easily seen from eqs. (2.43) and (2.44) that we must have

$$\left[(A_+v_- - A_-a_{--}e^{-2ij\pi})e^{3i\alpha_+} + (A_-v_+ + A_+a_{++}e^{2ij\pi})e^{3i\alpha_-}\right]e^{2\eta\pi i} =$$
$$\left[(B_+u_+ + B_-a_{--})e^{i\alpha_+} + (B_-u_- - B_+a_{++})e^{i\alpha_-}\right]e^{-i\delta}, \quad (2.45)$$
$$\left[(A_+v_- - A_-a_{--}e^{-2ij\pi})e^{5i\alpha_+} + (A_-v_+ + A_+a_{++}e^{2ij\pi})e^{5i\alpha_-}\right]e^{2\eta\pi i} =$$
$$\left[(B_+u_+ + B_-a_{--})e^{-i\alpha_+} + (B_-u_- - B_+a_{++})e^{-i\alpha_-}\right]e^{i\delta}. \quad (2.46)$$

Finally, we should rotate to the branch containing point B. This makes $z - \delta$ rotate through an angle of 2π, leading to the wave function

$$\Psi(z-\delta) \sim \left[(B_+v_- - B_-a_{--}e^{-2ij\pi})e^{5i\alpha_+} + (B_-v_+ + B_+a_{++}e^{2ij\pi})e^{5i\alpha_-}\right]\frac{e^{2\eta\pi i}e^{i(z-\delta)}}{2} +$$
$$\left[(B_+v_- - B_-a_{--}e^{-2ij\pi})e^{3i\alpha_+} + (B_-v_+ + B_+a_{++}e^{2ij\pi})e^{3i\alpha_-}\right]\frac{e^{2\eta\pi i}e^{-i(z-\delta)}}{2}.$$

Consequently, we have

$$\frac{(B_+v_- - B_-a_{--}e^{-2ij\pi})e^{5i\alpha_+} + (B_-v_+ + B_+a_{++}e^{2ij\pi})e^{5i\alpha_-}}{(A_+u_+ + A_-a_{--})e^{-i\alpha_+} + (A_-u_- - A_+a_{++})e^{-i\alpha_-}}$$
$$e^{2\eta\pi i}e^{-i\delta}e^{\frac{\pi\omega}{k_+} + \frac{\pi\omega}{k_C}} = \mathcal{M}(r_+)\mathcal{M}(r_C), \quad (2.47)$$

$$\frac{(B_+ v_- - B_- a_{--} e^{-2ij\pi})e^{3i\alpha_+} + (B_- v_+ + B_+ a_{++} e^{2ij\pi})e^{3i\alpha_-}}{(A_+ u_+ + A_- a_{--})e^{i\alpha_+} + (A_- u_- - A_+ a_{++})e^{i\alpha_-}}$$
$$e^{2\eta\pi i}e^{i\delta}e^{-\frac{\pi\omega}{k_H}-\frac{\pi\omega}{k_C}} = \mathcal{M}(r_+)\mathcal{M}(r_C). \quad (2.48)$$

The condition for these equations (2.45)-(2.48) to have nontrivial solutions (A_+, A_-, B_+, B_-) is then

$$\begin{vmatrix} s_1 e^{3i\alpha_+} & s_7 e^{3i\alpha_-} & -s_6 e^{i(\alpha_+ - \delta)} & -s_8 e^{i(\alpha_- - \delta)} \\ s_3 e^{5i\alpha_+} & s_5 e^{5i\alpha_-} & -s_2 e^{-i(\alpha_+ - \delta)} & -s_4 e^{-i(\alpha_- - \delta)} \\ s_2 e^{-i\alpha_+} e^{-\frac{2\pi\omega}{k_C}} & s_4 e^{-i\alpha_-} e^{-\frac{2\pi\omega}{k_C}} & -s_3 e^{i(5\alpha_+ - \delta)} & -s_5 e^{i(5\alpha_- - \delta)} \\ s_6 e^{i\alpha_+} e^{\frac{2\pi\omega}{k_+}} & s_8 e^{i\alpha_-} e^{\frac{2\pi\omega}{k_+}} & -s_1 e^{i(3\alpha_+ + \delta)} & -s_7 e^{i(3\alpha_- + \delta)} \end{vmatrix} = 0.$$

where

$$s_1 = (v_- + a_{++} e^{\frac{ij\pi}{2}})e^{2\eta\pi i}, \quad s_2 = u_+ - a_{++} e^{\frac{ij\pi}{2}},$$
$$s_3 = (v_- + a_{++} e^{-\frac{ij\pi}{2}})e^{2\eta\pi i}, \quad s_4 = u_- + a_{--} e^{-\frac{ij\pi}{2}},$$
$$s_5 = (v_+ - a_{--} e^{\frac{ij\pi}{2}})e^{2\eta\pi i}, \quad s_6 = u_+ - a_{++} e^{-\frac{ij\pi}{2}},$$
$$s_7 = (v_+ - a_{--} e^{-\frac{ij\pi}{2}})e^{2\eta\pi i}, \quad s_8 = u_- + a_{--} e^{\frac{ij\pi}{2}}.$$

After some algebra, we obtain

$$\mathcal{A}_1 \cosh\left(\frac{\pi\omega}{k_+} - \frac{\pi\omega}{k_C} - \Delta_1\right) + \mathcal{A}_2 \cosh\left(\frac{\pi\omega}{k_+} + \frac{2\pi\omega}{k_-} + \frac{\pi\omega}{k_C} + \Delta_2\right) + \mathcal{A}_3 \cosh\left(\frac{\pi\omega}{k_+} + \frac{\pi\omega}{k_C}\right) = 0,$$
$$(2.49)$$

where

$$\mathcal{A}_1 = \left[\left(2s_1 s_3 s_5 s_7 - s_3^2 s_7^2 e^{ij\pi} - s_1^2 s_5^2 e^{-ij\pi}\right)\left(2s_2 s_4 s_6 s_8 - s_4^2 s_6^2 e^{ij\pi} - s_2^2 s_8^2 e^{-ij\pi}\right)\right]^{\frac{1}{2}},$$
$$\mathcal{A}_2 = \left[\left(2s_3 s_5 s_6 s_8 - s_3^2 s_8^2 e^{2ij\pi} - s_5^2 s_6^2 e^{-2ij\pi}\right)\left(2s_1 s_2 s_4 s_7 - s_1^2 s_4^2 e^{2ij\pi} - s_2^2 s_7^2 e^{-2ij\pi}\right)\right]^{\frac{1}{2}},$$
$$\mathcal{A}_3 = s_3 s_4 s_6 s_7 e^{ij\pi} + s_1 s_2 s_5 s_8 e^{-ij\pi} - s_1 s_3 s_4 s_8 e^{2ij\pi} - s_2 s_5 s_6 s_7 e^{-2ij\pi},$$
$$\Delta_1 = \frac{1}{2}\left[\log\left(2s_1 s_3 s_5 s_7 - s_3^2 s_7^2 e^{ij\pi} - s_1^2 s_5^2 e^{-ij\pi}\right) - \log\left(2s_2 s_4 s_6 s_8 - s_4^2 s_6^2 e^{ij\pi} - s_2^2 s_8^2 e^{-ij\pi}\right)\right],$$
$$\Delta_2 = \frac{1}{2}\left[\log\left(2s_3 s_5 s_6 s_8 - s_3^2 s_8^2 e^{2ij\pi} - s_5^2 s_6^2 e^{-2ij\pi}\right) - \right.$$
$$\left. \log\left(2s_1 s_2 s_4 s_7 - s_1^2 s_4^2 e^{2ij\pi} - s_2^2 s_7^2 e^{-2ij\pi}\right)\right].$$

For zeroth-order asymptotic QN frequencies, one has $a_{\pm\pm} = 0$, and hence $u_\pm = 1$ and $v_\pm = e^{-2\eta\pi i}$. As a result, formula (2.49) reduces to (2.40).

As discussed in ref. [17], the Stokes lines for $d = 4$ and $d = 5$ are a bit different from the case discussed in the previous calculation for $d = 6$. On account of the behavior of $r \sim \infty$, they found that the final result for $d = 4$ is the same as eq. (2.49). However, the result for $d = 5$ is changed. For $r \sim \infty$, the coefficient of e^{-iz} in the formula of the wave function keeps unchanged, while the coefficient of e^{iz} in there reverses sign as one rotates from the branch containing point B to the branch containing point A in the contour. Therefore, one can complete this calculation by reversing the sign of eq. (2.47). In the end, we obtain

$$\mathcal{A}_1 \sinh\left(\frac{\pi\omega}{k_C} - \frac{\pi\omega}{k_+} + \Delta_1\right) + \mathcal{A}_2 \sinh\left(\frac{\pi\omega}{k_+} + \frac{2\pi\omega}{k_-} + \frac{\pi\omega}{k_C} + \Delta_2\right) + \mathcal{A}_3 \sinh\left(\frac{\pi\omega}{k_+} + \frac{\pi\omega}{k_C}\right) = 0,$$

For $d = 4$ RNdS black holes, one has $r_C \gg r_+ > r_-$ as $0 < \lambda \ll 1$. As a result, one has $-k_- > k_+ \gg k_C$. In this case, we can neglect $1/k_+$ and $-1/k_-$ compared to $1/k_C$. Taking eq. (2.40) into account, one finds the asymptotic QN frequencies in this limit is the solutions of this formula

$$e^{\frac{\pi\omega}{k_C}} + e^{-\frac{\pi\omega}{k_C}} = 0.$$

So, we have $\frac{\omega}{T_C} = (2n+1)\pi i$, where T_C is the Hawking temperature at cosmological horizon. However, as one lets λ approach its extremal value, i.e.,

$$\lambda \to \frac{2 + 2\sqrt{9 - 8q^2}}{(3 + \sqrt{9 - 8q^2})^3},$$

with $m = 1$. [35] showed that in this extremal case we have $k_H \approx k_C \to 0$, while k_- does not approach zero. So, in this limit, one can neglect the term with regard to $1/k_-$, and obtain the asymptotic QN frequencies by solving this formula

$$e^{2\frac{\pi\omega}{k_C}} + e^{-2\frac{\pi\omega}{k_C}} = 0,$$

which in turn derives $\frac{\omega}{T_C} = \frac{1}{2}(2n+1)\pi i$. This reminds us to conjecture that the frequencies have values between these two extremal values, i.e.,

$$\frac{\omega}{T_C} \simeq \chi(2n+1)\pi i + \mathrm{Re}\,\omega, \qquad \frac{1}{2} < \chi < 1, \qquad (2.50)$$

where χ is a parameter closely related to the cosmological constant λ. In fact, it would be an interesting work to investigate the relationship between these two parameters, by analytical methods or numerical ones. $\mathrm{Re}\,\omega$ is the real part of the frequency.

In order to obtain the explicit expressions of $\mathcal{A}_1$, $\mathcal{A}_2$, Δ_1 and Δ_2, we need the integral

$$\int_0^\infty dz\, z^{-2/3} J_\nu(z) J_\mu(z) = \frac{\Gamma(\frac{2}{3})\Gamma(\frac{\mu+\nu+1/3}{2})}{\sqrt[3]{4}\,\Gamma(\frac{\nu-\mu+5/3}{2})\Gamma(\frac{\mu+\nu+5/3}{2})\Gamma(\frac{\mu-\nu+5/3}{2})}.$$

As a result, we obtain

$$s_1 = 1 - 2U(j)\sin\frac{(5+3j)\pi}{6}, \qquad\qquad s_2 = 1 + (1 - \sqrt{3}i)U(j)\sin\frac{(5+3j)\pi}{6},$$

$$s_3 = 1 + (1 + \sqrt{3}i)U(j)\sin\frac{(5+3j)\pi}{6}, \quad s_4 = 1 + (1 - \sqrt{3}i)U(j)\sin\frac{(5-3j)\pi}{6},$$

$$s_5 = 1 + (1 + \sqrt{3}i)U(j)\sin\frac{(5-3j)\pi}{6}, \quad s_6 = 1 + 2U(j)\sin\frac{(5+3j)\pi}{6},$$

$$s_7 = 1 - 2U(j)\sin\frac{(5-3j)\pi}{6}, \qquad\qquad s_8 = 1 + 2U(j)\sin\frac{(5-3j)\pi}{6},$$

where

$$U(j) = \frac{W(j)\Gamma(2/3)\Gamma^2(1/6)\Gamma(1/6 + j/2)\Gamma(1/6 - j/2)}{16\pi^2 r_0^- \sqrt{(4n+2)\chi k_C}}.$$

After some algebra, we obtain the explicit expressions of $\mathcal{A}_1$, $\mathcal{A}_2$, Δ_1 and Δ_2 in terms of $o(1/\sqrt{n})$

$$\mathcal{A}_1 \simeq 2(1 - \cos j\pi)[1 + (1 + \sqrt{3}i)U(j)\cos j\pi/2],$$

$$\mathcal{A}_2 \simeq 2(1 - \cos 2j\pi) \left\{ 1 + \left[\cos j\pi/2 - (3/2 + \sqrt{3}i) \frac{\cos j\pi}{\cos j\pi/2} \right] U(j) \right\},$$

$$\mathcal{A}_3 \simeq 2(\cos j\pi - \cos 2j\pi) \left\{ 1 + \left[4 \cos j\pi/2 + 2\sqrt{3}i \frac{2\cos^2 j\pi - \cos j\pi}{1 + 2\cos j\pi} \right] U(j) \right\},$$

$$\Delta_1 \simeq (3\sqrt{3}i - 5) U(j) \cos j\pi/2,$$

$$\Delta_2 \simeq U(j) \left[(2 + \sqrt{3}i) \cos j\pi/2 + \frac{\sqrt{3}i}{2} \cdot \frac{\cos j\pi}{\cos j\pi/2} \right].$$

For vector type perturbations, we have $j = 5/3 = 2 - 1/3$. Therefore, as one inserts $j = 5/3$ into $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, Δ_1 and Δ_2, we have exactly the same results as in the $j = 1/3$ case(tensor or scalar type perturbations), and consequently eq. (2.49) becomes

$$\cosh \left(\frac{\pi\omega}{k_+} - \frac{\pi\omega}{k_C} + (5\sqrt{3}/2 - 3i) U(j) \right)$$

$$+ \left[3 - \frac{(3\sqrt{3} + 15i) U(j)}{2} \right] \cosh \left(\frac{\pi\omega}{k_+} + \frac{2\pi\omega}{k_-} + \frac{\pi\omega}{k_C} + (\sqrt{3} + 2i) U(j) \right)$$

$$+ [2 + (5\sqrt{3} + 3i) U(j)] \cosh \left(\frac{\pi\omega}{k_+} + \frac{\pi\omega}{k_C} \right) = 0, \qquad (2.51)$$

for three type perturbations. Inserting the expressions of $W(j)$ (see appendix A), we can easily obtain the explicit expression of $U(j)$. In this way, the first-order correction to asymptotic QN frequencies of RNdS black holes can be obtained by evaluating eq. (2.51). The results show that the correction term is closely related to ℓ, q, λ (though the parameter χ and k_C), and of course, n.

Numerical calculation of this case has not been performed in the previous literature. Consequently, it is interesting to perform numerical checks to our first-order corrected results both in four and higher dimensional black hole spacetime.

2.6 The Schwarzschild anti-de Sitter case

This case was studied analytically in [23]. Here we list their results for completeness. Considering the behavior of the black hole singularity $(r \sim 0)$ and large r, they obtained the QN frequencies for all type of perturbations,

$$\bar{z} = \frac{\pi}{4} (2 + j + j_\infty) - \tan^{-1} \frac{i}{1 + 2e^{ij\pi}} + n\pi,$$

where $j_\infty = d - 1, d - 3$ and $d - 5$ for tensor, vector and scalar perturbations, respectively, and $\bar{z} = \omega \bar{r}_*$ is the integration constant of the tortoise coordinate (refer the reader to appendix A).

By expanding the wave function to the first order in $1/\omega^{(d-3)/(d-2)}$ as

$$\Psi = \Psi^{(0)} + \frac{1}{\omega^{(d-3)/(d-2)}} \Psi^{(1)},$$

ref. [23] showed that the first-order correction to the QN frequency is

$$\bar{z} = \omega \bar{r}_* = \frac{\pi}{4} (2 + j + j_\infty) + \frac{\ln 2}{2i} + n\pi - \frac{1}{8} \left[6i\bar{b} - 2ie^{-i\pi \frac{d-3}{d-2}} \bar{b} - 9\bar{a}_1 + \bar{a}_2 + e^{-i\pi \frac{d-3}{d-2}} (\bar{a}_1 - \bar{a}_2) \right],$$

where

$$\bar{a}_1 = a_1(\infty), \quad \bar{a}_2 = a_2(\infty), \quad \bar{b} = b(\infty),$$

and $a_1(z), a_2(z), b(z)$ are defined as

$$a_1(z) = \frac{\pi W(j)}{8} \omega^{-\frac{d-3}{d-2}} \int_0^z dz' z'^{-\frac{1}{d-2}} J_{j/2}(z') J_{j/2}(z'),$$

$$a_2(z) = \frac{\pi W(j)}{8} \omega^{-\frac{d-3}{d-2}} \int_0^z dz' z'^{-\frac{1}{d-2}} N_{j/2}(z') N_{j/2}(z'),$$

$$b(z) = \frac{\pi W(j)}{8} \omega^{-\frac{d-3}{d-2}} \int_0^z dz' z'^{-\frac{1}{d-2}} J_{j/2}(z') N_{j/2}(z').$$

$W(j)$ in these formulae is defined as

$$W(j) = \begin{cases} W_{SAdST} & j = j_T, \\ W_{SAdSV} & j = j_V, \\ W_{SAdSS} & j = j_S, \end{cases}$$

and the explicit expressions of W_{SAdST}, W_{SAdSV} and W_{SAdSS} can be found in appendix A.

For $d = 4$, one obtains

$$\omega_n \bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) + n\pi + corr_4, \tag{2.52}$$

where

$$corr_4 = \frac{\ln 2}{2i} - \frac{(1+i)W(j)}{128\pi^2} \cdot \sqrt{\frac{\bar{r}_*}{n}} \left[\cot \frac{\pi}{2} \left(\frac{1}{2} - j \right) - 3 \right]$$

$$\times \left(\cos \frac{j\pi}{2} - \sin j\pi 2 \right) \Gamma^2(1/4) \Gamma(1/4 + j/2) \Gamma(1/4 - j/2),$$

and we have used the integral

$$\int_0^\infty dz z^{-1/2} J_\nu(z) J_\mu(z) = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{\mu+\nu+1/2}{2})}{\sqrt{2} \Gamma(\frac{\nu-\mu+3/2}{2}) \Gamma(\frac{\mu+\nu+3/2}{2}) \Gamma(\frac{\mu-\nu+3/2}{2})}.$$

In this way, one can easily obtain the explicit expression of $corr_4$ of eq. (2.52). [23] have given the explicit expressions for tensor, vector and scalar type perturbations in $d = 4$ case. Moreover, they showed that their results are in good agreement with the numerical results performed in [36].

2.7 The Reissner-Nordström anti-de Sitter case

We now compute analytically the quasinormal modes of the RN AdS d-dimensional black hole including first-order corrections. Previous work on this case can be found in [37 – 39]. Here we start with the zeroth-order calculation. For Schwarzschild Anti-de Sitter black hole, we have

$$f(r) = 1 - \frac{2m}{r^{d-3}} - \lambda r^2, \tag{2.53}$$

with the roots

$$r_n = r_+, r_- r_1, r_1^*, \dots, r_{d-3}, r_{d-3}^*,$$

where $\lambda(< 0)$ is the black hole background parameter related to the cosmological constant Λ by

$$\Lambda = \frac{1}{2}(d-1)(d-2)\lambda,$$

and r_n^* represents the conjugate of r_n. Near the black hole singularity ($r \sim 0$), the tortoise coordinate may be expanded as

$$r_* = \int \frac{dr}{f(r)} = \frac{1}{2d-5}\frac{r^{2d-5}}{q^2} + \frac{2m}{3d-8}\frac{r^{3d-8}}{q^4} + \cdots.$$

In this procedure, we have assumed $\frac{r}{r_0^-} \ll 1$, where $r_0^- = \left(m - \sqrt{m^2 - q^2}\right)^{\frac{1}{d-3}}$, represents the inner horizon of the RN black hole. Again we must expand the potential to the first order in $1/\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}$ instead of $1/\omega^{(d-3)/(2d-5)}$. After defining $z = \omega r_*$, the potential for the three different type perturbations can be expanded, respectively, as (appendix A)

$$V[z] \sim -\frac{\omega^2}{4z^2}\left\{1 - j^2 - W(j)\left(\frac{z}{(r_0^-)^{2d-5}\omega}\right)^{(d-3)/(d-2)} + \cdots\right\}, \qquad (2.54)$$

where

$$W(j) = \begin{cases} W_{RNAdST} & j = j_T, \\ W_{RNAdSV\pm} & j = j_{V\pm}, \\ W_{RNAdSS\pm} & j = j_{S\pm}, \end{cases}$$

and the explicit expressions of W_{RNAdST}, $W_{RNAdSV\pm}$ and $W_{RNAdSS\pm}$ can be found in appendix A. Then the Schrödinger-like wave equation (2.1) with the potential (2.54) can be depicted as

$$\left(\mathcal{H}_0 + \left[(r_0^-)^{2d-5}\omega\right]^{-\frac{d-3}{d-2}}\mathcal{H}_1\right)\Psi = 0, \qquad (2.55)$$
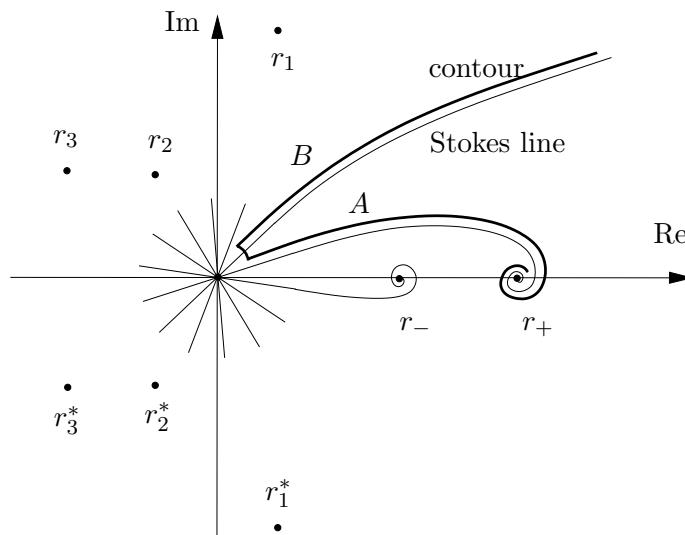
where $\mathcal{H}_0$ and $\mathcal{H}_1$ are defined as

$$\mathcal{H}_0 = \frac{d^2}{dz^2} + \left[\frac{1-j^2}{4z^2} + 1\right], \quad \mathcal{H}_1 = -\frac{W(j)}{4}z^{-\frac{3d-7}{2d-5}}.$$

Obviously, the zeroth-order wave equation can be written as

$$\mathcal{H}_0\Psi^{(0)} = 0,$$

with solutions in the form of Bessel function

$$\Psi^{(0)}(z) = \sqrt{\frac{\pi z}{2}}J_{j/2}(z) + A\sqrt{\frac{\pi z}{2}}N_{j/2}(z).$$

Figure 5: Stokes line for the Reissner-Nordström Anti-de Sitter black hole, along with the chosen contour for monodromy matching, in the case of dimension $d = 6$ (we refer the reader to [17] for detail, and a more complete list of figures in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$).

As one lets $z \to +\infty$, it behaves as

$$\Psi^{(0)}(z) \sim \left(e^{-i\alpha_+} - iAe^{-i\alpha_+}\right)\frac{e^{iz}}{2} + \left(e^{i\alpha_+} + iAe^{i\alpha_+}\right)\frac{e^{-iz}}{2}.$$

This holds at point B in figure 5. Appendix A shows that the potential for large r behaves as

$$V = \frac{j_\infty - 1}{4(z - \bar{z})^2} + \cdots$$

where $j_\infty = d - 1, d - 3$ and $d - 5$ for tensor, vector and scalar perturbations, respectively, and $\bar{z}$ is the integration constant of the tortoise coordinate

$$\bar{z} = \omega\bar{r}_* = \omega\int_0^\infty \frac{dr}{f(r)}.$$

Then the Schrödinger-like wave equation (2.1) with this potential can be depicted as

$$\frac{d^2\Psi_\infty(z)}{dz^2} + \left[\frac{1 - j_\infty^2}{4(z - \bar{z})^2} + 1\right]\Psi_\infty(z) = 0.$$

On account of the boundary condition $\Psi_\infty(z) \to 0$ as $r \to \infty$, one can obtain the asymptotic behavior of the wave function

$$\Psi_\infty(z) \sim \frac{B}{2}\left[e^{iz}e^{-i(\bar{z}+\beta)} + e^{-iz}e^{i(\bar{z}-\beta)}\right], \tag{2.56}$$

where $\beta = \frac{\pi}{4}(1 + j_\infty)$. Since on the same branch, we must let $\Psi^{(0)}(z) = \Psi_\infty(z)$, then one finds

$$A = \tan(\bar{z} - \beta - \alpha_+). \tag{2.57}$$

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As one rotates from the branch containing point B to the branch containing point A in the contour, through an angle of $\frac{-\pi}{2d-5}$, $z \sim \frac{\omega r^{2d-5}}{(2d-5)q^2}$ rotates through an angle of $-\pi$, leading to the wave function

$$\Psi^{(0)}(z) \sim (1 - iA) \frac{e^{-i\alpha_+} e^{iz}}{2} + \left[1 - iA\left(1 + 2e^{ij\pi}\right)\right] \frac{e^{-3i\alpha_+} e^{-iz}}{2},$$

as $z \to -\infty$.

From the boundary conditions (2.2), we have

$$1 - iA\left(1 + 2e^{ij\pi}\right) = 0.$$

Consequently, taking eq. (2.57) into account, one obtains the QN frequencies

$$\bar{z} = \frac{\pi}{4}(2 + j + j_\infty) - \tan^{-1} \frac{i}{1 + 2e^{ij\pi}} + n\pi$$

Next, we calculate the first-order correction to the asymptotic frequencies. Again we expand the wave function to the first order in $1/\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}$ as

$$\Psi = \Psi^{(0)} + \frac{1}{\left[(r_0^-)^{2d-5}\omega\right]^{(d-3)/(2d-5)}} \Psi^{(1)}.$$

Then one can rewrite eq. (2.55) as

$$\mathcal{H}_0\Psi^{(1)} + \mathcal{H}_1\Psi^{(0)} = 0.$$

The general solution of this equation is

$$\Psi^{(1)}(z) = \frac{\pi\sqrt{z}J_{j/2}(z)}{2} \int_0^z dz' \sqrt{z'} N_{j/2}(z') \mathcal{H}_1\Psi^{(0)}(z')$$
$$- \frac{\pi\sqrt{z}N_{j/2}(z)}{2} \int_0^z dz' \sqrt{z'} J_{j/2}(z') \mathcal{H}_1\Psi^{(0)}(z').$$

The behavior as $z \to +\infty$ is found to be

$$\Psi_\pm^{(1)}(z) \sim \left(\mathcal{A}_1 e^{-i\alpha_+} - i\mathcal{A}_2 e^{-i\alpha_+}\right) \frac{e^{iz}}{2} + \left(\mathcal{A}_1 e^{i\alpha_+} + i\mathcal{A}_2 e^{i\alpha_+}\right) \frac{e^{-iz}}{2}.$$

Here we defined

$$\mathcal{A}_1 = 1 - \bar{b} - A\bar{a}_2, \qquad \mathcal{A}_2 = (1 + \bar{b})A + \bar{a}_1,$$

where

$$\bar{a}_1 = a_1(\infty), \quad \bar{a}_2 = a_2(\infty), \quad \bar{b} = b(\infty),$$

and $a_1(z), a_2(z), b(z)$ are defined as

$$a_1(z) = \frac{\pi W(j)}{8} \left[(r_0^-)^{2d-5}\omega\right]^{-\frac{d-3}{2d-5}} \int_0^z dz' z'^{-\frac{d-2}{2d-5}} J_{j/2}(z') J_{j/2}(z'),$$
$$a_2(z) = \frac{\pi W(j)}{8} \left[(r_0^-)^{2d-5}\omega\right]^{-\frac{d-3}{2d-5}} \int_0^z dz' z'^{-\frac{d-2}{2d-5}} N_{j/2}(z') N_{j/2}(z'),$$

$$b(z) = \frac{\pi W(j)}{8} \left[(r_0^-)^{2d-5} \omega \right]^{-\frac{d-3}{2d-5}} \int_0^z dz' z'^{-\frac{d-2}{2d-5}} J_{j/2}(z') N_{j/2}(z').$$

This holds at point B in figure 5. For the same reason have been discussed above for the zeroth-order case, we should let $\Psi(z) = \Psi_\infty(z)$. Combining eq. (2.56), one finds that

$$\mathcal{A}_2 = \mathcal{A}_1 \tan(\bar{z} - \beta - \alpha_+)$$

as $z \to \infty$.

Again, as one rotates from the branch containing point B to the branch containing point A in the contour, through an angle of $\frac{-\pi}{2d-5}$, $z \sim \frac{\omega r^{2d-5}}{(2d-5)q^2}$ rotates through an angle of $-\pi$, leading to the wave function (as $z \to -\infty$)

$$\Psi^{(0)}(z) \sim \frac{C e^{iz}}{2} + \left\{ \left[1 - iA \left(1 + 2e^{ij\pi} \right) \right] e^{-3i\alpha_+} + e^{-i\alpha_+} e^{-i\pi\frac{d-3}{2d-5}} \left(\bar{B}_1 - i\bar{B}_2 \right) \right\} \frac{e^{-iz}}{2},$$

where

$$\bar{B}_1 = -2\cos\frac{j\pi}{2} \left[e^{-ij\pi} \bar{a}_1 + \left(A - e^{-2i\alpha_+} \right) \left[\bar{b} - i \left(1 + e^{-ij\pi} \right) \bar{a}_1 \right] \right] -$$
$$- Ae^{-2i\alpha_+} \left[e^{ij\pi} \bar{a}_2 - 4\cos^2\frac{j\pi}{2} \bar{a}_1 - 2i \left(1 + e^{ij\pi} \right) \bar{b} \right],$$
$$\bar{B}_2 = e^{-2i\alpha_-} \left[e^{-ij\pi} \bar{a}_1 + A \left[\bar{b} - i \left(1 + e^{-ij\pi} \right) \bar{a}_1 \right] \right],$$

and C is a constant that can be easily calculated but is not needed here.
Taking the boundary conditions (2.2) into account, we have

$$\left[1 - iA \left(1 + 2e^{ij\pi} \right) \right] e^{-3i\alpha_+} + e^{-i\alpha_+} e^{-i\pi\frac{d-3}{2d-5}} \left(\bar{B}_1 - i\bar{B}_2 \right) = 0.$$

Consequently, combining eq. (2.57), one obtains the QN frequencies

$$\bar{z} = \omega \bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) + \frac{\ln 2}{2i} + n\pi - \frac{1}{8} \left[6i\bar{b} - 2ie^{-i\pi\frac{d-3}{2d-5}} \bar{b} - 9\bar{a}_1 + \bar{a}_2 + e^{-i\pi\frac{d-3}{2d-5}} \left(\bar{a}_1 - \bar{a}_2 \right) \right]$$

For $d = 4$, one obtains

$$\omega_n \bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) - \tan^{-1}\frac{i}{1 + 2e^{ij\pi}} + n\pi + corr_4, \tag{2.58}$$

where

$$corr_4 = \frac{1}{2i} \ln(\cos j\pi/2) - \frac{(1 + \sqrt{3}i)W(j)}{1024\pi^2 r_0^-} \sqrt[3]{\frac{2\bar{r}_*}{n\pi}} \left[(5\sqrt{3} + 1) \sin j\pi/2 - (5 - \sqrt{3}) \cos j\pi/2 \right] \cdot$$
$$\cdot \Gamma(2/3)\Gamma^2(1/6)\Gamma(1/6 + j/2)\Gamma(1/6 - j/2),$$

and we have used the integral

$$\int_0^\infty dz z^{-2/3} J_\nu(z) J_\mu(z) = \frac{\Gamma(\frac{2}{3})\Gamma(\frac{\mu+\nu+1/3}{2})}{\sqrt[3]{4}\Gamma(\frac{\nu-\mu+5/3}{2})\Gamma(\frac{\mu+\nu+5/3}{2})\Gamma(\frac{\mu-\nu+5/3}{2})}.$$

For $d = 4$, the roots of the formula $f(r) = 0$ can be depicted by r_+ (the black hole outer horizon), r_- (the black hole inner horizon), and two complex roots, r_1 and its complex

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conjugate r_1^*. As discussed in [23], it is convenient to set $\lambda = -1$. From (2.53), the roots are

$$r_+ = \frac{1}{2}(g + \sqrt{-2 - g^2 + 4m/g}), \quad r_- = \frac{1}{2}(g - \sqrt{-2 - g^2 + 4m/g}),$$

$$r_1 = \frac{1}{2}(-g + \sqrt{-2 - g^2 - 4m/g}), \quad r_1^* = \frac{1}{2}(-g - \sqrt{-2 - g^2 - 4m/g}),$$

with $g = \sqrt{-\frac{2}{3} + \frac{h_+ + h_-}{3\sqrt{2}}}$, where

$$h_\pm = \left(2 - 72q^2 + 108m^2 \pm \sqrt{-4(1 + 12q^2)^3 + (2 - 72q^2 + 108m^2)^2}\right)^{\frac{1}{3}}.$$

The integration constant in the tortoise coordinate can be then solved as

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)}$$

$$= -\frac{r_-^2}{3r_-^3 + r_- - r_+ r_1 r_1^*} \ln \frac{r_-}{r_+} - \frac{r_1^2}{3r_1^3 + r_1 - r_+ r_- r_1^*} \ln \frac{r_1}{r_+} - \frac{r_1^{*2}}{3r_1^{*3} + r_1^* - r_+ r_- r_1} \ln \frac{r_1^*}{r_+}.$$

Obviously, $\bar{r}_*$ is the formula (105) in [23] as $q = 0$ (or the Schwarzschild limit).

In this way, one can easily obtain the value of QN frequencies in eq. (2.58). For tensor type perturbations, $W_{RNAdST} = 0$, so $corr_4 = -i\frac{\ln 3}{4}$. However, it seems unavailable for scalar type perturbations, since in this case we have $j \to 1/3$, which may induce the integral $\int_0^\infty dz z^{-2/3} J_{-1/3}(z) J_{-1/3}(z)$ approaches infinity. It is interesting to investigate this problem. Is there any other methods can avoid this singularity? For vector type perturbations , we have $j \to 5/3$. This leads to

$$corr_4 = -i\frac{\ln 3}{4} + \frac{(15 - \sqrt{3})(1 + \sqrt{3}i)W_{RNAdSV\pm}}{1024\pi r_0^-} \sqrt[3]{\frac{2\bar{r}_*}{n\pi}} \cdot \Gamma^2(1/6),$$

where

$$W_{RNAdSV\pm} = \frac{4}{3}\left(m \pm \sqrt{9m^2 + 4\ell(\ell+1)q^2 - 8q^2}\right)(1 - \sqrt{1 - q^2})(3q^2)^{-2/3}$$

From this formula we find: (1) in order to insure $r_0^-(4n + 2)^{1/3} \gg 1$ as one calculates the QN frequencies of the first-order correction, the imaginary part of the frequencies (or the modes n) needs bigger values for a black hole with small charge, since $r_0^- \sim q^2 \to 0$ as $q \to 0$. This confirms the prediction made by Neitzke in [30]: the required n diverges as $q \to 0$, and the corrections would blow up this divergence; (2) the first-order correction to the asymptotic QN frequencies are shown to be dimension dependent and related closely to m, ℓ, j, and the charge q.

QN frequencies of RN Ads black holes were first calculated numerically by Berti and Kokkotas in [40], in the case of $d = 4$. [41] latter performed an extensive numerical study of QN frequencies for massless scalar fields in $d = 4$ RN AdS spacetime. They find for the higher modes, both Re ω and Im ω increase with n. As q increases, Im ω increases faster than Re ω does, with n. [17] matched these predictions to their zeroth-order analytical

formula of the asymptotic QN frequency. Their results are complete agreement with the numerical calculations of [41]. In our first-order corrected formula, the frequencies can be obtained by subtracting $i\frac{\ln 3}{4}$ from their results. This leads, obviously, to the same gap of ω as [17] does. Although more complete numerical results are needed to check our analytical results in higher dimensional cases, our results are in good agreement with the numerical results in the particular case of $d = 4$.

3. Conclusions and future directions

In this paper we studied analytically quasinormal modes in a wide variety of black hole spacetimes, including d-dimensional asymptotically flat spacetimes and non-asymptotically flat spacetimes. We extend the procedure of [17] to include first-order corrections to analytical expressions for QN frequencies by making use of a monodromy technique, which was first introduced in [16] for zeroth-order approximation, and later extended to first-order for Schwarzschild black hole in [22]. The calculation performed in this paper shows that systematic expansions for uncharged black holes include different corrections with the ones for charged black holes. In other words, d-dimensional uncharged black holes have an expansion including corrections in $1/\omega^{(d-2)/(d-3)}$, while charged ones have an expansion in $1/\omega^{(d-3)/(2d-5)}$. This difference makes them have a different n-dependence relation in the first-order correction formulae.

The method applied above in calculating the first-order corrections of QN frequencies seems to be unavailable for black holes with small charge, since the required n diverges as the charge $q \to 0$, and worse, the corrections would blow this divergence up. More extensive investigation is needed for this problem. A very recent work on RN black holes appeared in [42], where they discussed a possible way to avoid this divergence for zeroth-order case in the small charge limit.

Some remarks are due for the particular case of $d = 4$ dimensional spacetimes, as in these cases we discussed more deeply. For Schwarzschild case, we obtained a j and ℓ-dependence correction term, which was proved by numerical results. For RN case, a puzzle appears when we apply our method to calculate the first-order corrections of the scalar type perturbations, since the correction in this case approaches infinity. In fact, this problem also exists in other four dimensional charged black holes. Strangely enough, extremal RN black hole can avoid this problem. Moreover, it seems that the correction term equals to zero, independently of the type perturbations. An investigation on whether the first-order corrections for any $d > 3$ extremal RN black holes have this behavior, should be further performed. Another point worthy of being mentioned about RN case is that the extremal limits ($q \to 0$, or $q \to m$) of the RN QN frequencies do not yield the right QN frequencies in their extremal cases (Schwarzschild QN frequencies, or extremal RN QN frequencies). Our results show that the same thing happens in the first-order corrections. [43] found that the limit is a singular one since it involves topology change at the level of the contours in the complex plane.

On the applications to quantum gravity [17] indicate that the $\ln 3$ in $d = 4$ Schwarzschild seems to be nothing but some numerical coincidences. Our results support this

argument. However, a recent paper on quantization of charged black holes [44] found that other than the Schwarzschild case, the asymptotic resonance corrgoesponds to a fundamental area unit $\Delta A = 4\hbar \ln 2$. According to Dreyer's conjecture [13], one can immediately set the spin network unit as $1/2$, i.e., $j_{min} = 1/2$. This differs from $j_{min} = 1$ for Schwarzschild case. From the LQG point of view, this result supports the claims [14] that the gauge group of LQG should be SU(2) in spite of the $\ln 3$ for Schwarzschild. Is there any other explanation of that? Or more generally, can we find other similar proof to support Dreyer's conjecture for other spacetimes?

There are some other possible directions for further study: (1) perform some numerical calculations to check our analytical solutions done above; (2) make sure if it is possible to have such an easy relation between the asymptotic QN frequencies and the overtone modes n as suggested in the asymptotically dS cases(2.30) and (2.50), or more deeply, find the possible relation between χ and λ and a possible explanation of that; (3) find an explanation of our correction results on the physical side.

Acknowledgements

A. The expanded master equation potentials

In this appendix we shall deduce the expressions of the potentials in Schrödinger-like equation (2.1). Natário and Schiappa [17] have deduced the value of the potentials in the region of large r. For completeness, we may list them in this appendix. Our work focuses on the region of the origin, which plays an key role in our procedure to calculate the first-order correction of the asymptotic QN frequencies using monodromy method.

We start with a set of potentials determined by the type of perturbations and can be deduced from a set of master equations which were derived by Ishibashi and Kodama in $[24-26]$ (can be denoted as the Ishibashi-Kodama master equations). Making use of these potentials, we make an exhaustive study of the first-order values of all these potentials in terms of the tortoise coordinate. The zeroth-order values of these potentials can be found in [17]. For tensor, vector and scalar perturbations, we obtain, respectively,

$$V_{\mathsf{T}}(r) = f(r)\left(\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right). \qquad (A.1)$$

$$V_{\mathsf{V}\pm}(r) = f(r)\left(\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} - \frac{(d-1)m}{r^{d-1}} + \frac{(d-2)^2 q^2}{r^{2d-4}} \pm \frac{\Delta}{r^{d-1}}\right), \qquad (A.2)$$

where

$$\Delta \equiv \sqrt{(d-1)^2 (d-3)^2 m^2 + 2 (d-2) (d-3) \Big(\ell (\ell + d - 3) - (d-2) \Big) q^2}. \tag{A.3}$$

$$V_{\mathsf{S}\pm}(r) = \frac{f(r) U_\pm(r)}{64 r^2 H_\pm^2(r)}. \tag{A.4}$$

The expressions of $U_\pm$ and $H_\pm$ can be found in [17]. In these expressions, the Ψ_+ equation represents the electromagnetic mode and the Ψ_- equation represents the gravitational mode.

The Schwarzschild case: In this case, the black hole is uncharged, so one has $V_{\mathsf{V}+} = V_{\mathsf{V}-}$ and $V_{\mathsf{S}+} = V_{\mathsf{S}-}$. For convenience, we denote them by V_{V} and V_{S}, respectively. The background space-time metric is

$$f(r) = 1 - \frac{2m}{r^{d-3}}.$$

Near the black hole singularity ($r \sim 0$), the tortoise coordinate can be expanded as

$$r_* = \int \frac{dr}{f(r)} = -\frac{1}{d-2} \frac{r^{d-2}}{2m} - \frac{1}{2d-5} \frac{r^{2d-5}}{(2m)^2} + \cdots,$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$. Using this formula, one may obtain an approximate expression of r in terms of the tortoise coordinate r_*

$$r \sim [-2m(d-2)r_*]^{\frac{1}{d-2}} + \frac{(d-2)r_*}{2d-5}. \tag{A.5}$$

As one applies (A.5) to the three different potentials (eqs. (A.1), (A.2) and (A.4)), these potentials can be expanded near the black hole singularity in terms of the tortoise coordinate, respectively, as

$$V_{\mathsf{T}}(r) \sim -\frac{1}{4z^2} \left[1 - 0^2 - W_{ST} \left(\frac{z}{\omega} \right)^{\frac{d-3}{d-2}} \right]$$

$$V_{\mathsf{V}}(r) \sim -\frac{1}{4z^2} \left[1 - 2^2 - W_{SV} \left(\frac{z}{\omega} \right)^{\frac{d-3}{d-2}} \right]$$

$$V_{\mathsf{S}}(r) \sim -\frac{1}{4z^2} \left[1 - 0^2 - W_{SS} \left(\frac{z}{\omega} \right)^{\frac{d-3}{d-2}} \right],$$

where

$$W_{ST} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}} \left[\frac{2(d-3)^2}{2d-5} + \frac{4\ell(\ell+d-3)}{d-2} \right],$$

$$W_{SV} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}} \left[\frac{2(d^2 - 8d + 13)}{2d - 15} + \frac{4\ell(\ell+d-3)}{d-2} \right],$$

$$W_{SS} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}} \left[\frac{2d^3 - 24d^2 + 94d - 116}{(2d-5)(d-2)} \right.$$

$$+ \frac{4(d^2 - 7d + 14)[\ell(\ell + d - 3) - d + 2]}{(d-1)(d-2)^2} \Bigg],$$

and we have rescaled the tortoise coordinate $z = \omega r_*$.

Near $r = \infty$ one has

$$V_{\mathsf{T}}(r) \sim 0,$$
$$V_{\mathsf{V}}(r) \sim 0,$$
$$V_{\mathsf{S}}(r) \sim 0.$$

The RN case: The background space-time metric is

$$f(r) = 1 - \frac{2m}{r^{d-3}} + \frac{q^2}{r^{2d-6}}.$$

Near the black hole singularity ($r \sim 0$), the tortoise coordinate can be expanded as

$$r_* = \int \frac{dr}{f(r)} = \frac{1}{2d-5} \frac{r^{2d-5}}{q^2} + \frac{2m}{3d-8} \frac{r^{3d-8}}{q^4} + \cdots .$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$, and we have assumed $r/r_0^- \ll 1$. Using this formula, one may obtain an approximate expression of r in terms of the tortoise coordinate r_*

$$\frac{r}{r_0^-} \sim [(2d-5)q^2(r_0^-)^{5-2d} r_*]^{\frac{1}{2d-5}} - \frac{2m(r_0^-)^{d-3}}{(3d-8)q^2}[(2d-5)q^2(r_0^-)^{5-2d} r_*]^{\frac{d-2}{2d-5}}. \qquad (A.6)$$

As one applies (A.6) to the five different potentials (eqs. (A.1), (A.2) and (A.4)), these potentials can be expanded near the black hole singularity in terms of the tortoise coordinate, respectively, as

$$V_{\mathsf{T}}(r) \sim -\frac{1}{4z^2} \left[1 - j_T^2 - W_{RNT} \left(\frac{z}{(r_0^-)^{2d-5}\omega} \right)^{\frac{d-3}{2d-5}} \right]$$

$$V_{\mathsf{V}\pm}(r) \sim -\frac{1}{4z^2} \left[1 - j_{V\pm}^2 - W_{RNV\pm} \left(\frac{z}{(r_0^-)^{2d-5}\omega} \right)^{\frac{d-3}{2d-5}} \right]$$

$$V_{\mathsf{S}\pm}(r) \sim -\frac{1}{4z^2} \left[1 - j_{S\pm}^2 - W_{RNS\pm} \left(\frac{z}{(r_0^-)^{2d-5}\omega} \right)^{\frac{d-3}{2d-5}} \right],$$

where

$$j_T = j_{S\pm} = \frac{d-3}{2d-5}, \qquad j_{V\pm} = \frac{3d-7}{2d-5} = 2 - j_T,$$

and we have rescaled the tortoise coordinate $z = \omega r_*$. The expressions of W_{RNT}, $W_{RNV\pm}$ and $W_{RNS\pm}$ in these equations are

$$W_{RNT} = 0,$$
$$W_{RNV\pm} = \frac{4m(d-2)(d^2 - d - 4) - 4(3d-8)[(d-1)m \mp \Delta]}{(2d-5)(3d-8)} (r_0^-)^{d-3} \left[(2d-5)q^2 \right]^{\frac{2-d}{2d-5}},$$

$$W_{RNS\pm} = -[2m(d-2)(4d-10)q^2 + \mathcal{W}_\pm](r_0^-)^{d-3}\left[(2d-5)q^2\right]^{\frac{7-3d}{2d-5}},$$

where

$$\mathcal{W}_+ = -mq^2[-(d-1)(3d-8)(1-\Omega) + 2(4d^2 - 15d + 12)] - 2(d-2)(3d-8)mq^2 -$$
$$-\frac{4(3d-8)[\ell(\ell+d-3) - (d-2)]q^4}{(d-1)(1-\Omega)m},$$
$$\mathcal{W}_- = -mq^2[(d-1)(3d-8)(1-\Omega) + 2(d-2)^2] - 2(d-2)(3d-8)mq^2 -$$
$$-\frac{4(3d-8)[\ell(\ell+d-3) - (d-2)]q^4}{(d-1)(1+\Omega)m},$$

with the definition, here and below, that

$$\Omega = \sqrt{1 + \frac{4[\ell(\ell+d-3) - d + 2]q^2}{(d-1)^2 m^2}}.$$

For the extremal case, $q \to m$, we have

$$V_{\mathsf{T}}(r) \sim -\frac{1}{4z^2}\left[1 - j_T^2 - W_{RNT}^{ex}\left(\frac{z}{\omega}\right)^{\frac{d-3}{2d-5}}\right]$$
$$V_{\mathsf{V}\pm}(r) \sim -\frac{1}{4z^2}\left[1 - j_{V\pm}^2 - W_{RNV\pm}^{ex}\left(\frac{z}{\omega}\right)^{\frac{d-3}{2d-5}}\right]$$
$$V_{\mathsf{S}\pm}(r) \sim -\frac{1}{4z^2}\left[1 - j_{S\pm}^2 - W_{RNS\pm}^{ex}\left(\frac{z}{\omega}\right)^{\frac{d-3}{2d-5}}\right],$$

where

$$W_{RNT}^{ex} = 0,$$
$$W_{RNV\pm}^{ex} = \frac{4m(d-2)(d^2-d-4) - 4(3d-8)[(d-1)m \mp \Delta^{ex}]}{(2d-5)(3d-8)}\left[(2d-5)m^2\right]^{\frac{2-d}{2d-5}},$$
$$W_{RNS\pm}^{ex} = 0,$$

and

$$\Delta^{ex} = \sqrt{(d-1)^2(d-3)^2 + 2(d-2)(d-3)\left(\ell(\ell+d-3) - (d-2)\right)}m.$$

Near $r = \infty$ one finds

$$V_{\mathsf{T}}(r) \sim 0,$$
$$V_{\mathsf{V}}(r) \sim 0,$$
$$V_{\mathsf{S}}(r) \sim 0.$$

The Schwarzschild dS case: In this case, the black hole is uncharged, so one has $V_{\mathsf{V}+} = V_{\mathsf{V}-}$ and $V_{\mathsf{S}+} = V_{\mathsf{S}-}$. For convenience, we denote them by V_{V} and V_{S}, respectively. The background space-time metric is

$$f(r) = 1 - \frac{2m}{r^{d-3}} - \lambda r^2,$$

with $\lambda > 0$. Near the black hole singularity ($r \sim 0$), the tortoise coordinate can be expanded as

$$r_* = \int \frac{dr}{f(r)} = -\frac{1}{d-2}\frac{r^{d-2}}{2m} - \frac{1}{2d-5}\frac{r^{2d-5}}{(2m)^2} + \cdots ,$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$. Using this formula, one may obtain an approximate expression of r in terms of the tortoise coordinate r_*

$$r \sim [-2m(d-2)r_*]^{\frac{1}{d-2}} + \frac{(d-2)r_*}{2d-5}. \tag{A.7}$$

As one applies (A.7) to the three different potentials (eqs. (A.1), (A.2) and (A.4)), these potentials can be expanded near the black hole singularity in terms of the tortoise coordinate, respectively, as

$$V_{\mathsf{T}}(r) \sim -\frac{1}{4z^2}\left[1 - 0^2 - W_{SdST}\left(\frac{z}{\omega}\right)^{\frac{d-3}{d-2}}\right]$$

$$V_{\mathsf{V}}(r) \sim -\frac{1}{4z^2}\left[1 - 2^2 - W_{SdSV}\left(\frac{z}{\omega}\right)^{\frac{d-3}{d-2}}\right]$$

$$V_{\mathsf{S}}(r) \sim -\frac{1}{4z^2}\left[1 - 0^2 - W_{SdSS}\left(\frac{z}{\omega}\right)^{\frac{d-3}{d-2}}\right],$$

where

$$W_{SdST} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}}\left[\frac{2(d-3)^2}{2d-5} + \frac{4\ell(\ell+d-3)}{d-2}\right],$$

$$W_{SdSV} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}}\left[\frac{2(d^2-8d+13)}{2d-15} + \frac{4\ell(\ell+d-3)}{d-2}\right],$$

$$W_{SdSS} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}}\left[\frac{2d^3-24d^2+94d-116}{(2d-5)(d-2)}\right.$$

$$\left. +\frac{4(d^2-7d+14)[\ell(\ell+d-3)-d+2]}{(d-1)(d-2)^2}\right],$$

and we have rescaled the tortoise coordinate $z = \omega r_*$. It is easily seen that this is just like in the pure Schwarzschild case.

Near $r = \infty$ one finds

$$r_*[r] \sim -\frac{1}{\lambda}\int \frac{dr}{r^2} = \bar{r}_* + \frac{1}{\lambda r},$$

which leads to

$$V_{\mathsf{T}}(r) \sim \frac{d(d-2)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-1)^2-1]}{4(z-\bar{z})^2},$$

$$V_{\mathsf{V}}(r) \sim \frac{(d-2)(d-4)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-3)^2-1]}{4(z-\bar{z})^2},$$

$$V_{\mathsf{S}}(r) \sim \frac{(d-4)(d-6)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-5)^2-1]}{4(z-\bar{z})^2},$$

as one defines $z = \omega r_*$ and $\bar{z} = \omega \bar{r}_*$.

The RN dS case: The background space-time metric is

$$f(r) = 1 - \frac{2m}{r^{d-3}} + \frac{q^2}{r^{2d-6}} - \lambda r^2,$$

with $\lambda > 0$. Near the black hole singularity ($r \sim 0$), the tortoise coordinate can be expanded as

$$r_* = \int \frac{dr}{f(r)} = \frac{1}{2d-5}\frac{r^{2d-5}}{q^2} + \frac{2m}{3d-8}\frac{r^{3d-8}}{q^4} + \cdots .$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$, and we have assumed $r/r_0^- \ll 1$ ($r_0^- = \left(m - \sqrt{m^2 - q^2}\right)^{\frac{1}{d-3}}$ represents the inner horizon of the RN black hole). Using this formula, one may obtain an approximate expression of r in terms of the tortoise coordinate r_*

$$\frac{r}{r_0^-} \sim [(2d-5)q^2(r_0^-)^{5-2d}r_*]^{\frac{1}{2d-5}} - \frac{2m(r_0^-)^{d-3}}{(3d-8)q^2}[(2d-5)q^2(r_0^-)^{5-2d}r_*]^{\frac{d-2}{2d-5}}. \tag{A.8}$$

As one applies (A.8) to the five different potentials (eqs. (A.1), (A.2) and (A.4)), these potentials can be expanded near the black hole singularity in terms of the tortoise coordinate, respectively, as

$$V_{\mathsf{T}}(r) \sim -\frac{1}{4z^2}\left[1 - j_T^2 - W_{RNdST}\left(\frac{z}{(r_0^-)^{2d-5}\omega}\right)^{\frac{d-3}{2d-5}}\right]$$

$$V_{\mathsf{V}\pm}(r) \sim -\frac{1}{4z^2}\left[1 - j_{V\pm}^2 - W_{RNdSV\pm}\left(\frac{z}{(r_0^-)^{2d-5}\omega}\right)^{\frac{d-3}{2d-5}}\right]$$

$$V_{\mathsf{S}\pm}(r) \sim -\frac{1}{4z^2}\left[1 - j_{S\pm}^2 - W_{RNdSS\pm}\left(\frac{z}{(r_0^-)^{2d-5}\omega}\right)^{\frac{d-3}{2d-5}}\right],$$

where

$$j_T = j_{S\pm} = \frac{d-3}{2d-5}, \quad j_{V\pm} = \frac{3d-7}{2d-5} = 2 - j_T,$$

and we have rescaled the tortoise coordinate $z = \omega r_*$. The expression of W_{RNdST}, $W_{RNdSV\pm}$ and $W_{RNdSS\pm}$ in these equations are

$$W_{RNdST} = 0,$$

$$W_{RNdSV\pm} = \frac{4m(d-2)(d^2-d-4) - 4(3d-8)[(d-1)m \mp \Delta]}{(2d-5)(3d-8)}(r_0^-)^{d-3}\left[(2d-5)q^2\right]^{\frac{2-d}{2d-5}},$$

$$W_{RNdSS\pm} = -[2m(d-2)(4d-10)q^2 + \mathcal{W}_\pm](r_0^-)^{d-3}\left[(2d-5)q^2\right]^{\frac{7-3d}{2d-5}},$$

where

$$\mathcal{W}_+ = -mq^2[-(d-1)(3d-8)(1-\Omega) + 2(4d^2-15d+12)] - 2(d-2)(3d-8)mq^2 - $$
$$-\frac{4(3d-8)[\ell(\ell+d-3) - (d-2)]q^4}{(d-1)(1-\Omega)m},$$

$$\mathcal{W}_- = -mq^2[(d-1)(3d-8)(1-\Omega) + 2(d-2)^2] - 2(d-2)(3d-8)mq^2 - $$

$$-\frac{4(3d-8)[\ell(\ell+d-3)-(d-2)]q^4}{(d-1)(1+\Omega)m}.$$

This is just like the pure RN case.

Near $r=\infty$ one finds

$$r_*[r] \sim -\frac{1}{\lambda}\int\frac{dr}{r^2} = \bar{r}_* + \frac{1}{\lambda r},$$

which leads to

$$V_{\mathsf{T}}(r) \sim \frac{d(d-2)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-1)^2-1]}{4(z-\bar{z})^2},$$

$$V_{\mathsf{V}}(r) \sim \frac{(d-2)(d-4)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-3)^2-1]}{4(z-\bar{z})^2},$$

$$V_{\mathsf{S}}(r) \sim \frac{(d-4)(d-6)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-5)^2-1]}{4(z-\bar{z})^2},$$

as one defines $z = \omega r_*$ and $\bar{z} = \omega \bar{r}_*$.

The Schwarzschild AdS case: In this case, the black hole is uncharged, so one has $V_{\mathsf{V}+} = V_{\mathsf{V}-}$ and $V_{\mathsf{S}+} = V_{\mathsf{S}-}$. For convenience, we denote them by V_{V} and V_{S}, respectively. The background space-time metric is

$$f(r) = 1 - \frac{2m}{r^{d-3}} - \lambda r^2,$$

with $\lambda < 0$. Near the black hole singularity ($r \sim 0$), the tortoise coordinate can be expanded as

$$r_* = \int\frac{dr}{f(r)} = -\frac{1}{d-2}\frac{r^{d-2}}{2m} - \frac{1}{2d-5}\frac{r^{2d-5}}{(2m)^2} + \cdots,$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$. Using this formula, one may obtain an approximate expression of r in terms of the tortoise coordinate r_*

$$r \sim [-2m(d-2)r_*]^{\frac{1}{d-2}} + \frac{(d-2)r_*}{2d-5}. \tag{A.9}$$

As one applies (A.9) to the three different potentials (eqs. (A.1), (A.2) and (A.4)), these potentials can be expanded near the black hole singularity in terms of the tortoise coordinate, respectively, as

$$V_{\mathsf{T}}(r) \sim -\frac{1}{4z^2}\left[1-0^2 - W_{SAdST}\left(\frac{z}{\omega}\right)^{\frac{d-3}{d-2}}\right]$$

$$V_{\mathsf{V}}(r) \sim -\frac{1}{4z^2}\left[1-2^2 - W_{SAdSV}\left(\frac{z}{\omega}\right)^{\frac{d-3}{d-2}}\right]$$

$$V_{\mathsf{S}}(r) \sim -\frac{1}{4z^2}\left[1-0^2 - W_{SAdSS}\left(\frac{z}{\omega}\right)^{\frac{d-3}{d-2}}\right],$$

where

$$W_{SAdST} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}}\left[\frac{2(d-3)^2}{2d-5} + \frac{4\ell(\ell+d-3)}{d-2}\right],$$

$$W_{SAdSV} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}} \left[\frac{2(d^2 - 8d + 13)}{2d - 15} + \frac{4\ell(\ell + d - 3)}{d - 2} \right],$$

$$W_{SAdSS} = \frac{1}{[-2(d-2)m]^{\frac{1}{d-2}}} \left[\frac{2d^3 - 24d^2 + 94d - 116}{(2d - 5)(d - 2)} \right.$$

$$\left. + \frac{4(d^2 - 7d + 14)[\ell(\ell + d - 3) - d + 2]}{(d - 1)(d - 2)^2} \right],$$

and we have rescaled the tortoise coordinate $z = \omega r_*$. It is easily seen that this is just like in the pure Schwarzschild case.

Near $r = \infty$ one finds

$$r_*[r] \sim -\frac{1}{|\lambda|} \int \frac{dr}{r^2} = \bar{r}_* + \frac{1}{|\lambda| r},$$

which leads to

$$V_{\mathsf{T}}(r) \sim \frac{d(d-2)}{4(r_* - \bar{r}_*)^2} = \frac{\omega[(d-1)^2 - 1]}{4(z - \bar{z})^2},$$

$$V_{\mathsf{V}}(r) \sim \frac{(d-2)(d-4)}{4(r_* - \bar{r}_*)^2} = \frac{\omega[(d-3)^2 - 1]}{4(z - \bar{z})^2},$$

$$V_{\mathsf{S}}(r) \sim \frac{(d-4)(d-6)}{4(r_* - \bar{r}_*)^2} = \frac{\omega[(d-5)^2 - 1]}{4(z - \bar{z})^2},$$

as one defines $z = \omega r_*$ and $\bar{z} = \omega \bar{r}_*$.

The RN AdS case: The background space-time metric is

$$f(r) = 1 - \frac{2m}{r^{d-3}} + \frac{q^2}{r^{2d-6}} - \lambda r^2,$$

with $\lambda < 0$. Near the black hole singularity ($r \sim 0$), the tortoise coordinate can be expanded as

$$r_* = \int \frac{dr}{f(r)} = \frac{1}{2d - 5} \frac{r^{2d-5}}{q^2} + \frac{2m}{3d - 8} \frac{r^{3d-8}}{q^4} + \cdots.$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$, and we have assumed $r/r_0^- \ll 1$ ($r_0^- = \left(m - \sqrt{m^2 - q^2}\right)^{\frac{1}{d-3}}$ represents the inner horizon of the RN black hole). Using this formula, one may obtain an approximate expression of r in terms of the tortoise coordinate r_*

$$\frac{r}{r_0^-} \sim [(2d - 5)q^2(r_0^-)^{5-2d}r_*]^{\frac{1}{2d-5}} - \frac{2m(r_0^-)^{d-3}}{(3d - 8)q^2}[(2d - 5)q^2(r_0^-)^{5-2d}r_*]^{\frac{d-2}{2d-5}}. \tag{A.10}$$

As one applies (A.10) to the five different potentials (eqs. (A.1), (A.2) and (A.4)), these potentials can be expanded near the black hole singularity in terms of the tortoise coordinate, respectively, as

$$V_{\mathsf{T}}(r) \sim -\frac{1}{4z^2} \left[1 - j_T^2 - W_{RNAdST} \left(\frac{z}{(r_0^-)^{2d-5}\omega} \right)^{\frac{d-3}{2d-5}} \right]$$

$$V_{\mathsf{V}\pm}(r) \sim -\frac{1}{4z^2}\left[1 - j_{V\pm}^2 - W_{RNAdSV\pm}\left(\frac{z}{(r_0^-)^{2d-5}\omega}\right)^{\frac{d-3}{2d-5}}\right]$$

$$V_{\mathsf{S}\pm}(r) \sim -\frac{1}{4z^2}\left[1 - j_{S\pm}^2 - W_{RNAdSS\pm}\left(\frac{z}{(r_0^-)^{2d-5}\omega}\right)^{\frac{d-3}{2d-5}}\right],$$

where

$$j_T = j_{S\pm} = \frac{d-3}{2d-5}, \quad j_{V\pm} = \frac{3d-7}{2d-5} = 2 - j_T,$$

and we have rescaled the tortoise coordinate $z = \omega r_*$. The expression of W_{RNAdST}, $W_{RNAdSV\pm}$ and $W_{RNAdSS\pm}$ in these equations are

$$W_{RNAdST} = 0,$$
$$W_{RNAdSV\pm} = \frac{4m(d-2)(d^2-d-4) - 4(3d-8)[(d-1)m \mp \Delta]}{(2d-5)(3d-8)}(r_0^-)^{d-3}\left[(2d-5)q^2\right]^{\frac{2-d}{2d-5}},$$
$$W_{RNAdSS\pm} = -[2m(d-2)(4d-10)q^2 + \mathcal{W}_\pm](r_0^-)^{d-3}\left[(2d-5)q^2\right]^{\frac{7-3d}{2d-5}},$$

where

$$\mathcal{W}_+ = -mq^2[-(d-1)(3d-8)(1-\Omega) + 2(4d^2-15d+12)] - 2(d-2)(3d-8)mq^2 -$$
$$-\frac{4(3d-8)[\ell(\ell+d-3)-(d-2)]q^4}{(d-1)(1-\Omega)m},$$
$$\mathcal{W}_- = -mq^2[(d-1)(3d-8)(1-\Omega) + 2(d-2)^2] - 2(d-2)(3d-8)mq^2 -$$
$$-\frac{4(3d-8)[\ell(\ell+d-3)-(d-2)]q^4}{(d-1)(1+\Omega)m}.$$

This is just like the pure RN case.

Near $r = \infty$ one finds

$$r_*[r] \sim -\frac{1}{|\lambda|}\int \frac{dr}{r^2} = \bar{r}_* + \frac{1}{|\lambda| r},$$

which leads to

$$V_{\mathsf{T}}(r) \sim \frac{d(d-2)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-1)^2-1]}{4(z-\bar{z})^2},$$
$$V_{\mathsf{V}}(r) \sim \frac{(d-2)(d-4)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-3)^2-1]}{4(z-\bar{z})^2},$$
$$V_{\mathsf{S}}(r) \sim \frac{(d-4)(d-6)}{4(r_*-\bar{r}_*)^2} = \frac{\omega[(d-5)^2-1]}{4(z-\bar{z})^2},$$

as one defines $z = \omega r_*$ and $\bar{z} = \omega \bar{r}_*$.

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